6143

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This proves the first assertion of the problem.

Now $Q(s,r)>2$ implies $\log p_r > 3.623(1+\delta)\log p_t$ or

$$p_r > p_t^{3.623+\varepsilon}, \qquad \varepsilon = 3.623\delta = o(1). \tag{8}$$

Let p_r be the least prime satisfying (8), then we infer from (4) and (5)

$$r-1=\pi\left(p_t^{3.623+\varepsilon}\right)<\frac{1.255}{3.623+\varepsilon}p_t^{3.623+\varepsilon}(\log p_t)^{-1}$$
$$<\frac{1.255}{3.623+\varepsilon}t^{3.623+\varepsilon}(\log t+\log\log t)^{2.623+\varepsilon}.$$

Putting $s=n$, $r=n+k$ we get

$$k<t\left[\frac{1.255}{3.623+\varepsilon}\{t\log(t\log t)\}^{2.623+\varepsilon}-1\right], \qquad t=n-1. \tag{9}$$

The table of the exact values of k begins as follows

n	1	2	3	4	5	6	7	8	9	10	11
k	3	5	11	21	35	61	75	97	130	167	204

In this range, (9) with $\varepsilon=-1$ gives a reasonable upper bound for $n\geqslant 5$.

Also solved by David Anderson, Charles Cable, Robert Dressler, Ernst Heppner & Wolfgang Schwarz (Germany), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), James Ridley (South Africa), David Rusin, and Manuel Scarowsky (Canada),

Notes. (1) Rusin's calculations lead to the asymptotic result: minimum value of $k \sim -n+(e/2)\,n^2\log n$.

(2) Calculation by Heppner and Schwarz yields the theorem: Let be $0<\varepsilon\leqslant 0.85$, and $n_0 \geqslant \max(100,\ 0.75254\log(1+\varepsilon/2),\ -0.64096\log(1-\varepsilon/2))$. Then $a_{n,k}$ is abundant if $n\geqslant n_0$ and $k\geqslant n^{2+\varepsilon}$, and not abundant if $n\geqslant n_0$ and $k\leqslant n^{2-\varepsilon}$. Here $a_{n,k}=\prod_{i=n+1}^{n+k}p_i$. For example, if $\varepsilon=0.85$ one may choose $n_0=2036$, and for $\varepsilon=0.5$ one may choose $n_0=2\cdot 10^7$.

Functions with Prescribed Discontinuities

6142 [1977, 222]. *Proposed by L. O. Chung, North Carolina State University*

Find a function $f:[0,1]\to[0,1]$ which is continuous everywhere except on two countable dense subsets D_1, D_2 of rationals such that on D_1 f is right continuous but not left continuous, and on D_2 f is left continuous but not right continuous.

Solution by Ellen Hertz, Paramount Design Company, New York City. Let D_1, D_2 be any pair of disjoint sets of rationals that are dense in $(0,1)$. Enumerate the elements of D_1 as $a_1, a_2, \ldots$, and the elements of D_2 as $b_1, b_2, \cdots$. Let $s_1, s_2, \ldots$, be a series of positive terms such that $\sum s_k = 1/2$.

For $0\leqslant x\leqslant 1$, set $f_1(x)=\sum_{n:a_n\leqslant x}s_n$. Then $f_1(0)=0$, $f_1(1)=1/2$, and f_1 is continuous except on D_1 where it is right continuous. Set $f_2(x)=\sum_{n:b_n<x}s_n$. Then f_2 is left continuous on D_2, continuous elsewhere. Then $f=f_1+f_2$ is as required.

Also solved by George Akst, Kenneth Andersen, Charles Belna, Eric Chandler, Michael Ecker, T. E. Gantner, Marguerite Gerstell, Eric Grinberg, Gustaf Gripenberg (Finland), I. I. Kotlarski, Joel Levy, O. P. Lossers (Netherlands), J. G. Mauldon, Gene Ortner, N. Fowler, Walter Stromquist, University of Wyoming Problem Group, and the proposer.

Note. For functions of the nature required, Gantner and Ortner refer us to Hewitt and Stromberg, *Real and Abstract Analysis*, p. 113; Andersen and Belna refer to Rudin, *Principles of Mathematical Analysis*, p. 84.

Dividing the Pie Fairly

6143 [1977, 222]. *Proposed by A. L. Macdonald, Eastern Michigan University*

The familiar method of fair division of a pie by passing a knife over it until someone is satisfied suggests the problem: Let $\pi_1, \pi_2, \ldots, \pi_n$ be non-atomic probability measures on a set X. Then there are pairwise disjoint sets $B_1, B_2, \ldots, B_n$ with $\pi_i(B_i) \geqslant 1/n$.

Solution by T. Sekiguchi, University of Arkansas. The proof is by induction: for $n=1$, the result is obvious. Assume the result for some $n \geqslant 1$.

Let $\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1}$ be $n+1$ non-atomic measures on X. By the induction hypothesis, there exists a partition $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n$ such that

$$\pi_i(\tilde{B}_i) \geqslant 1/n \quad \text{for } i=1,2,\ldots,n.$$

Now partition each $\tilde{B}_i$ into $B_{i1}, B_{i2}, \ldots, B_{i,n+1}$ with equal probabilities with respect to measure π_i, that is

$$\pi_i(B_{ij}) = \frac{1}{n+1}\pi_i(\tilde{B}_i), \qquad i=1,2,\ldots,n; \qquad j=1,2,\ldots,n+1.$$

Now arrange the second subindex with B_{i1} such that

$$\pi_{n+1}(B_{i1}) = \max_{1<j<n+1} \pi_{n+1}(B_{ij}), \qquad i=1,2,\ldots,n.$$

Define

$$B_i = \bigcup_{j=2}^{n+1} B_{ij}, \qquad i=1,2,\ldots,n$$

and

$$B_{n+1} = \bigcup_{i=1}^{n} B_{i1}.$$

For

$$1 \leqslant i \leqslant n, \qquad \pi_i(B_i) = \sum_{i=2}^{n+1} \pi_i(B_{ij}) = n\frac{1}{n+1}\pi_i(\tilde{B}_i) \geqslant \frac{1}{n+1}$$

and

$$\pi_{n+1}(B_{n+1}) = \sum_{i=1}^{n} \pi_{n+1}(B_{i1}) \geqslant \sum_{i=1}^{n} \frac{1}{n+1}\pi_{n+1}(\tilde{B}_i) = \frac{1}{n+1}\pi_{n+1}\left(\bigcup_{i=1}^{n} \tilde{B}_i\right) = \frac{1}{n+1}.$$

Now induction is complete.

Also solved by Leslie Arnold, Ethan Bolker, David Cantor, L. E. Clarke (England), Robert Field & Martin Ortel, Ellen Hertz, O. P. Lossers, (Netherlands), J. G. Mauldon, R. M. Norton, Henry Ricardo, Andrew Siegel, Stanford Statistics Problem Solving Group, J. G. Wendel, and the proposer.

Notes. (1) Mauldon shows with an example the necessity of the assumption that the measures π_i have the same family of measurable sets.

(2) The fair division problem and some recent references appear in A. M. Fink, *A note on the fair division problem*, Mathematics Magazine, vol. 37, p. 341.

(3) Bolker shows that there is a partition for $\pi_j(B_i) = 1/n$, $i=1,2,\ldots,n$; $j=1,2,\ldots,n$ so that "not only is each person satisfied, but each considers the whole partition fair."

(4) Ricardo points out that a generalization appears as corollaries 1.1, 1.2 in Dubins and Spanier, this MONTHLY, vol. 68, 1961, pp. 1 ff.

(5) Thurmon Whitley writes that the problem is a special case of Lemma 2 of *Relations among certain ranges of vector measures*, a paper by A. Dvoretsky, A. Wild, and J. Wolfowitz, which appeared in the Pacific Journal of Mathematics, 1951, p. 66. In fact, this lemma actually shows that the "greater than or equal" required in Problem 6143 can actually be "equality." Related results also appear in Whitley's Master's Thesis, "Some applications of vector-valued measures," University of North Carolina, 1966.