This proves the first assertion of the problem.  
Now \( Q(s,r) > 2 \) implies \( \log p_\epsilon > 3.623(1 + \delta) \log p_\delta \) or 
\[
\log p_\epsilon > p_\delta^{3.623 + \epsilon}, \quad \epsilon = 3.623 \delta = o(1). \tag{8}
\]
Let \( p_\epsilon \) be the least prime satisfying (8), then we infer from (4) and (5)
\[
r - 1 = \pi(p_\epsilon^{3.623 + \epsilon}) < \frac{1.255}{3.623 + \epsilon} p_\delta^{3.623 + \epsilon} (\log p_\delta)^{-1} \\
< \frac{1.255}{3.623 + \epsilon} (\log t + \log \log t)^{2.623 + \epsilon}.
\]
Putting \( s = n, \quad r = n + k \) we get
\[
k < \left[ \frac{1.255}{3.623 + \epsilon} (\log(\log t))^2 (2.623 + \epsilon - 1) \right], \quad t = n - 1. \tag{9}
\]
The table of the exact values of \( k \) begins as follows

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>21</td>
<td>35</td>
<td>61</td>
<td>75</td>
<td>97</td>
<td>130</td>
<td>167</td>
<td>204</td>
</tr>
</tbody>
</table>

In this range, (9) with \( \epsilon = -1 \) gives a reasonable upper bound for \( n > 5 \).

Also solved by David Anderson, Charles Cable, Robert Dressler, Ernst Heppner & Wolfgang Schwarz (Germany), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), James Ridley (South Africa), David Rusin, and Manuel Scarowsky (Canada).

Notes. (1) Rusin’s calculations lead to the asymptotic result: minimum value of \( k \sim -n + (e/2) n^2 \log n \).

(2) Calculation by Heppner and Schwarz yields the theorem: Let be \( 0 < e < 0.85 \), and \( n_0 > \max(100, 0.75254 \log(1 + e/2), -0.64096 \log(1 - e/2)) \). Then \( a_{n,k} \) is abundant if \( n > n_0 \) and \( k > n^2 + e \), and not abundant if \( n > n_0 \) and \( k < n^2 - e \). Here \( a_{n,k} = \left\lceil \frac{n^2 + k}{n + 1} \right\rceil \). For example, if \( e = 0.85 \) one may choose \( n_0 = 2036 \), and for \( e = 0.5 \) one may choose \( n_0 = 2 \cdot 10^7 \).

Functions with Prescribed Discontinuities


Find a function \( f : [0,1] \rightarrow [0,1] \) which is continuous everywhere except on two countable dense subsets \( D_1, D_2 \) of rationals such that on \( D_1 \) \( f \) is right continuous but not left continuous, and on \( D_2 \) \( f \) is left continuous but not right continuous.

Solution by Ellen Hertz, Paramount Design Company, New York City. Let \( D_1, D_2 \) be any pair of disjoint sets of rationals that are dense in \( (0,1) \). Enumerate the elements of \( D_1 \) as \( a_1, a_2, \ldots \), and the elements of \( D_2 \) as \( b_1, b_2, \cdots \). Let \( s_1, s_2, \ldots \) be a series of positive terms such that \( \sum s_k = \frac{1}{2} \).

For \( 0 < x < 1 \), set \( f_1(x) = \sum_{n:b_n < x} s_n \). Then \( f_1(0) = 0, f_1(1) = 1/2 \), and \( f_1 \) is continuous except on \( D_1 \) where it is right continuous. Set \( f_2(x) = \sum_{n:a_n < x} s_n \). Then \( f_2 \) is left continuous on \( D_2 \), continuous elsewhere. Then \( f = f_1 + f_2 \) as is required.


Note. For functions of the nature required, Gantner and Ortners refer us to Hewitt and Stromberg, Real and Abstract Analysis, p. 113; Andersen and Belna refer to Rudin, Principles of Mathematical Analysis, p. 84.

Dividing the Pie Fairly


The familiar method of fair division of a pie by passing a knife over it until someone is satisfied
suggests the problem: Let \( \pi_1, \pi_2, \ldots, \pi_n \) be non-atomic probability measures on a set \( X \). Then there are pairwise disjoint sets \( B_1, B_2, \ldots, B_n \) with \( \pi_i(B_i) > 1/n \).

Solution by T. Sekiguchi, University of Arkansas. The proof is by induction: for \( n = 1 \), the result is obvious. Assume the result for some \( n > 1 \).

Let \( \pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1} \) be \( n + 1 \) non-atomic measures on \( X \). By the induction hypothesis, there exists a partition \( \bar{B}_1, \bar{B}_2, \ldots, \bar{B}_n \) such that

\[
\pi_i(\bar{B}_i) > 1/n \quad \text{for } i = 1, 2, \ldots, n.
\]

Now partition each \( \bar{B}_i \) into \( B_{1i}, B_{2i}, \ldots, B_{ni}, B_{n+1i} \) with equal probabilities with respect to measure \( \pi_i \), that is

\[
\pi_i(B_{ji}) = \frac{1}{n+1}, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n+1.
\]

Now arrange the second subindex with \( B_{11} \) such that

\[
\pi_{n+1}(B_{11}) = \max_{1 < j < n+1} \pi_{n+1}(B_{j1}), \quad i = 1, 2, \ldots, n.
\]

Define

\[
B_i = \bigcup_{j=2}^{n+1} B_{ji}, \quad i = 1, 2, \ldots, n
\]

and

\[
B_{n+1} = \bigcup_{i=1}^{n} B_{1i}.
\]

For

\[
1 < i < n, \quad \pi_i(B_i) = \sum_{i=2}^{n+1} \pi_i(B_{ji}) = n \frac{1}{n+1} \pi_i(\bar{B}_i) > \frac{1}{n+1}
\]

and

\[
\pi_{n+1}(B_{n+1}) = \sum_{i=1}^{n} \pi_{n+1}(B_{1i}) \geq \sum_{i=1}^{n} \frac{1}{n+1} \pi_{n+1}(\bar{B}_i) = \frac{1}{n+1} \pi_{n+1} \left( \bigcup_{i=1}^{n} \bar{B}_i \right) = \frac{1}{n+1}.
\]

Now induction is complete.


Notes. (1) Mauldon shows with an example the necessity of the assumption that the measures \( \pi_i \) have the same family of measurable sets.


(3) Bolker shows that there is a partition for \( \pi_j(B_i) = 1/n, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n \) so that “not only is each person satisfied, but each considers the whole partition fair.”

(4) Ricardo points out that a generalization appears as corollaries 1.1, 1.2 in Dubins and Spanier, this MONTHLY, vol. 68, 1961, pp. 1 ff.

(5) Thurmon Whitley writes that the problem is a special case of Lemma 2 of *Relations among certain ranges of vector measures*, a paper by A. Dvoretsky, A. Wild, and J. Wolfowitz, which appeared in the Pacific Journal of Mathematics, 1951, p. 66. In fact, this lemma actually shows that the “greater than or equal” required in Problem 6143 can actually be “equality.” Related results also appear in Whitley’s Master’s Thesis, “Some applications of vector-valued measures,” University of North Carolina, 1966.