Potentials, Fields, and Sources with Geometric Calculus

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Abstract

Geometric calculus offers significant advantages over other formalisms in the treatment of potentials, fields, and sources. We show this for 3D Euclidean space and 4D Minkowski space. As a corollary we show that charge conservation implies the existence of a field satisfying Maxwell's equations.

One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity." – J. W. Gibbs

The relationships between sources, fields, and potentials is a fundamental issue in physics. Vector calculus is usually used to express the relationships in 3D Euclidean space. Vector calculus, tensor calculus, or differential forms are usually used in 4D Minkowski space. We show here that in both cases the realtionships are expressed and proved in a much simpler manner using geometric calculus. (I assume that the reader is familiar with the elements of geometric algebra and calculus [1].)

I also point out that charge conservation implies the existence of an electromagnetic field satisfying Maxwell's equations. The closest result to this of which I am aware is that charge conservation plus two of Maxwell's equations imply the other two equations.[3] Thus *charge conservation and the Lorentz force law encompass all of classical electromagnetism.*[1]

[1] Several years after this was written, Jose Heras obtained this result using vector calculus. 3D version: *Can Maxwell's equations be obtained from the continuity equation?*, Am. J. Phys. **75** 652 (2007). 4D version: *How to obtain the covariant form of Maxwell's equations from the continuity equation*, Eur. J. Phys. **30** 845 (2009).

Theorem 1 (3D Vector Calculus). *Let a scalar field s and a vector field $\mathbf{s}$ be time independent sources which vanish outside a bounded region R. Suppose that*

$$\nabla \cdot \mathbf{s} = 0. \tag{1}$$

Then the equations

$$\nabla \cdot \mathbf{f} = s, \tag{2}$$

$$\nabla \times \mathbf{f} = \mathbf{s} \tag{3}$$

have a unique vector field solution $\mathbf{f}$ which vanishes at infinity. The field $\mathbf{f}$ can be obtained from a scalar potential p and a vector potential $\mathbf{p}$:

$$\mathbf{f} = \nabla p - \nabla \times \mathbf{p}. \tag{4}$$

Note that Eq. (1) is necessary for Eq. (3) to have a solution, as the divergence of a curl is 0.

In vector calculus the relationships between sources, fields, and potentials are given by Eqs. (2)–(4). In geometric calculus the sources s and $\mathbf{s}$ are united into a single multivector source S, the gradient and curl are united into the geometric calculus derivative ∇, and the potentials p and $\mathbf{p}$ are united into a single multivector potential P. We shall see that ∇ expresses the relationships between potentials, fields, and sources in geometric calculus in the simplest possible way:

$$\nabla P = F \text{ and } \nabla F = S.$$

These two geometric calculus equations replace the vector calculus Eqs. (2)–(4).

The results here are closely related to Helmholtz's theorem [2]. Both involve potentials, fields, and sources related as above. The difference is of perspective: in the results here the source is given, whereas with Helmholtz's theorem the field is given.

Instead of proving Theorem 1 directly, we first prove the geometric calculus version of the theorem. We then prove Theorem 1 as a corollary.

Theorem 2 (3D Geometric Calculus). *Let S be a time independent bounded multivector source which vanishes outside a bounded region R. Then there is a unique multivector solution F to the equation*

$$\nabla F = S \tag{5}$$

satisfying $\lim_{x\to\infty} F(x) = 0$. *The solution satisfies* $|F(x)| \leq M/|x|^2$ *for some constant* M.

The field F has a potential P: $F = \nabla P$. The potential has the same grades as S. If S is a vector and $\nabla \cdot S = 0$, then $\nabla \cdot P = 0$.

Proof. Let s be a scalar field satisfying the conditions of the theorem. Then there is a unique scalar solution p to Poisson's equation $\nabla^2 p = s$ which vanishes at infinity [4]. The potential p is given by

$$p(x) = -\int_R \frac{s(x')}{4\pi|x-x'|}\, dV'.$$

Apply this grade-by-grade to S in Eq. (5) to obtain a multivector potential P with the same grades as S satisfying $\nabla^2 P = S$:

$$P(x) = -\int_R \frac{S(x')}{4\pi|x-x'|}\, dV'. \tag{6}$$

It is easy to see that $F = \nabla P$ is a solution to Eq. (5):

$$\nabla F = \nabla(\nabla P) = (\nabla\nabla)P = \nabla^2 P = S. \tag{7}$$

Since $\nabla(1/|x|) = -x/|x|^3$, Eq. (6) gives

$$F(x) = \nabla P(x) = -\int_R \nabla \frac{S(x')}{4\pi|x-x'|}\, dV' = \int_R \frac{(x-x')S(x')}{4\pi|x-x'|^3}\, dV'. \tag{8}$$

Eq. (8) gives an explicit solution of Eq. (5).

From Eq. (8),

$$|F(x)| \leq \int_R \frac{|S(x')|}{4\pi|x-x'|^2}\, dV'. \tag{9}$$

Since S is bounded and R is bounded, $|F(x)| \leq M/|x|^2$.

Now suppose that F_1 and F_2 are two solutions of Eq. (5) with $\lim_{x\to\infty} F_i(x) = 0$. Set $\Phi = F_2 - F_1$. Then $\lim_{x\to\infty} \Phi(x) = 0$. Moreover, Φ is harmonic: $\nabla^2\Phi = \nabla(\nabla F_2 - \nabla F_1) = 0$. According to the mean value theorem for harmonic functions, $\Phi(x) = (4\pi a^2)^{-1} \int_{|x'-x|=a} \Phi(x')\, dA'$. Let $a \to \infty$. Then $|\Phi(x)| \leq \sup_{|x'-x|=a} |\Phi(x')| \to 0$. This proves uniqueness.

Appendix 1 shows that if S is a vector with $\nabla \cdot S = 0$, then $\nabla \cdot P = 0$. $\square$

Theorem 2 is a good example of the clarity that geometric calculus can bring. In vector calculus the potential is obtained as above. But then the power of geometric calculus comes to the fore.

In geometric calculus ∇ can operate on all multivectors, in particular on vectors. This allows the trivial geometric calculus calculation Eq. (7), giving the simple relationships $\nabla P = F$ and $\nabla F = S$. Note the essential use of the associativity of the geometric product in Eq. (7): $\nabla(\nabla P) = (\nabla\nabla)P$. In vector calculus ∇^2 cannot be factored and $\nabla\mathbf{f}$ cannot be written for a vector $\mathbf{f}$. This leaves us with the more awkward relationships, Eqs. (2)–(4), between sources, fields, and potentials.

We can now prove Theorem 1. Of course this is of interest only for those continuing to use vector calculus instead of switching to geometric calculus.

Proof. Recall the geometric calculus identities

$$\nabla\mathbf{f} = \nabla\cdot\mathbf{f} + \nabla\wedge\mathbf{f} = \nabla\cdot\mathbf{f} - (\nabla\times\mathbf{f})^*. \tag{10}$$

Take the dual of Eq. (3), subtract from Eq. (2), and use Eq. (10), giving

$$\nabla\mathbf{f} = s - \mathbf{s}^*. \tag{11}$$

The single geometric calculus Eq. (11), with the stipulation that $\mathbf{f}$ be a vector, is equivalent to the two vector calculus Eqs. (2) and (3). From Theorem 2, Eq. (11) always has a geometric calculus solution $\mathbf{f}$, although it need not be a vector.

Eq. (6) shows that if $\mathbf{p}$ is the potential for the vector $\mathbf{s}$, then $\mathbf{p}^*$ is the potential for $\mathbf{s}^*$. Thus $\mathbf{f} = \nabla(p - \mathbf{p}^*)$, where p is a scalar and $\mathbf{p}^*$ is a bivector.

Theorem 1 shows that if $\nabla\cdot\mathbf{s} = 0$ (Eq. (1)), then $\nabla\cdot\mathbf{p} = 0$. And by the associativity of the geometric product, $\nabla(\mathbf{p}^*) = (\nabla\mathbf{p})^*$. Eq. (4) follows:

$$\mathbf{f} = \nabla(p - \mathbf{p}^*) = \nabla p - (\nabla\mathbf{p})^* = \nabla p - (\nabla\wedge\mathbf{p})^* = \nabla p - \nabla\times\mathbf{p}\,. \tag{12}$$

In particular, $\mathbf{f}$ is a vector. $\square$

Theorem 3 below treats time dependent sources, fields, and potentials. Its proof closely follows the proof of Theorem 2, except that it is based on solutions of the wave equation rather than on Poisson's equation. Standard proofs of related theorems use tensor analysis or differential forms [5]. They are complicated, in part because they are teeming with coordinate indices. Instead, we use again the coordinate free Eq. (7) of geometric calculus.

Theorem 3 (4D Geometric Calculus). *Let ∇ be the spacetime geometric calculus derivative: $\nabla = e_0\partial_t - e_1\partial_x - e_2\partial_y - e_3\partial_z$. Let S be a time dependent bounded multivector source which satisfies this condition: given (t,x) there is an $a>0$ so that $S(t-|x-x'|,x')=0$ for $|x-x'|\geq a$.*[2] *Then there is a multivector solution F to the equation*

$$\nabla F = S. \tag{13}$$

The field F has a potential P: $F=\nabla P$. The potential has the same grades as S. If S is a 4-vector with $\nabla \cdot S = 0$ then $\nabla \cdot P = 0$.

Proof. Note that $\nabla^2 = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$. Let s be a scalar field satisfying the conditions of the theorem. Then the wave equation $\nabla^2 p = s$ has a (retarded) solution [6]:

$$p(t,x) = -\int_{|x-x'|\leq a} \frac{s(t-|x-x'|,x')}{4\pi|x-x'|}\, dV'. \tag{14}$$

As in Eqs. (6)–(8) we obtain a solution to Eq. (13):

$$F = \nabla P(t,x) = -\int_{|x-x'|\leq a} \nabla\, \frac{S(t-|x-x'|,x')}{4\pi|x-x'|}\, dV'. \tag{15}$$

Appendix 2 shows that if S is a vector with $\nabla \cdot S = 0$, then $\nabla \cdot P = 0$. $\square$

Maxwell's four equations in vector calculus are expressed by a single equation in geometric calculus, $\nabla F = S$, where F is the electromagnetic field and S is the charge-current 4-vector.[3] This is the simplest equation expressing the charge as some sort of a derivative of the field. According to Theorem 3, Maxwell's equation has a solution $F = \nabla P$ given by Eq. (15). Moreover, $\nabla \cdot P = 0$, since $\nabla \cdot S = 0$ expresses charge conservation. Thus

$$F = \nabla P = \nabla \cdot P + \nabla \wedge P = \nabla \wedge P. \tag{16}$$

Geometric calculus represents an electromagnetic field with a bivector, $\nabla \wedge P$.

We have assumed charge conservation and proved the existence of a field satisfying Maxwell's equation:

Charge conservation $\implies$ *Maxwell's equation.*

This shows that there is no more physics in Maxwell's equation than in charge conservation. The additional physics in Maxwell's theory comes from the Lorentz force law.

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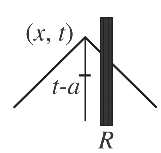
[2] The condition says that $S=0$ on the past light cone of (t,x) at sufficiently early times. The spacetime diagram shows that it is satisfied if $S(t,x)=0$ outside a fixed spatially bounded region R for all t. It is also satisfied if $S(t,x)=0$ outside a spatially bounded region at some time $t_0 \leq t$ and always moves with a velocity bounded away from $c=1$.



[3] Two equations are required in tensor calculus and differential forms [7].

Appendix 1. We complete the proof of Theorem 2 by showing that if S is a vector and $\nabla \cdot S = 0$, then $\nabla \cdot P = 0$. Compute

$$\begin{aligned}\nabla \cdot \frac{S(x')}{|x-x'|} &= \left(\nabla \frac{1}{|x-x'|}\right) \cdot S(x') \\ &= -\left(\nabla' \frac{1}{|x-x'|}\right) \cdot S(x') - \frac{1}{|x-x'|}\,\nabla' \cdot S(x') = -\nabla' \cdot \frac{S(x')}{|x-x'|}.\end{aligned} \tag{17}$$

Now use Eq. (6), Eq. (17), the divergence theorem, and $S|_{\partial R} = 0$:

$$\begin{aligned}\nabla \cdot P(x) &= -\int_R \nabla \cdot \frac{S(x')}{4\pi|x-x'|}\, dV' \\ &= \int_R \nabla' \cdot \frac{S(x')}{4\pi|x-x'|}\, dV' = \int_{\partial R} \frac{S(x')}{4\pi|x-x'|} \cdot dA' = 0.\end{aligned} \tag{18}$$

Appendix 2. We complete the proof of Theorem 3 by showing that if S is a vector and $\nabla \cdot S = 0$, then $\nabla \cdot P = 0$.

Let ∂_i be the partial derivative with respect to the i^{th} argument, $i = 0, 1, 2, 3$. For simplicity of notation in obtaining Eq. (19) below, we use only one spatial coordinate x. Then $S(t, x) = s_0(t, x)e_0 + s_1(t, x)e_1$. We may substitute anything we like for (t, x) in $\nabla \cdot S(t, x) = \partial_0 s_0(t, x) + \partial_1 s_1(t, x) = 0$. In particular,

$$\partial_0 s_0(t - |x-x'|, x') + \partial_1 s_1(t - |x-x'|, x') = 0.$$

Since only the 0^{th} argument of $S(t - |x-x'|, x')$ depends on t or x,

$$\begin{aligned}&\nabla \cdot S(t - |x-x'|, x') \\ &\quad = \partial_0 s_0(t - |x-x'|, x')\, \partial_t(t - |x-x'|) + \partial_0 s_1(t - |x-x'|, x')\, \partial_x(t - |x-x'|) \\ &\quad = -\partial_1 s_1(t - |x-x'|, x') + \partial_0 s_1(t - |x-x'|, x')\, \partial_{x'}|x-x'| \\ &\quad = -\nabla_3' \cdot S_3(t - |x-x'|, x'),\end{aligned} \tag{19}$$

where S_3 and ∇_3' are the spatial components of S and ∇'. From this,

$$\begin{aligned}&\nabla \cdot \frac{S(t - |x-x'|, x')}{|x-x'|} \\ &\quad = \left(\nabla \frac{1}{|x-x'|}\right) \cdot S(t - |x-x'|, x') + \frac{1}{|x-x'|}\, \nabla \cdot S(t - |x-x'|, x') \\ &\quad = \left(-\nabla_3' \frac{1}{|x-x'|}\right) \cdot S_3(t - |x-x'|, x') - \frac{1}{|x-x'|}\, \nabla_3' \cdot S_3(t - |x-x'|, x') \\ &\quad = -\nabla_3' \cdot \frac{S_3(t - |x-x'|, x')}{|x-x'|}.\end{aligned} \tag{20}$$

Now calculate as in Eq. (18) to obtain $\nabla \cdot P = 0$.

References

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