This paper is a continuation of [9].

1. Dual pairs of v.f.s.'s

We say that \((S(E), T(F))\) is a dual pair of v.f.s.'s if \(S(E)\) and \(T(F)\) are v.f.s.'s and if for every \(f \in S(E)\) and \(g \in T(F)\), \(\int \langle f, g \rangle \, d\pi\) exists. The integral \(\int \langle f, g \rangle \, d\pi\) will always be understood to be the bilinear form connecting the two spaces. Since \(T(F) \supseteq \Gamma(F)\), [9, Proposition 6.2] shows that \(T(F)\) separates points of \(S(E)\). Since \(S(E) \supseteq \Gamma(E)\) it is easy to show that \(S(E)\) separates points of \(T(F)\).

If \(S(E)\) is a v.f.s., the Köthe dual of \(S(E)\), denoted \(S(E)^*\), is the set of all \(g \in (E')\) such that \(\int \langle f, g \rangle \, d\pi\) exists for all \(f \in S(E)\). Since clearly \(S(E)^* \supseteq \Gamma(E')\), \(S(E)^*\) is a v.f.s. If \(T(F)\) is a v.f.s., \(T(F)^*\) is defined similarly and is a v.f.s.

There is a lack of symmetry in the concept of a dual pair of v.f.s.'s since \(S(E)\) and \(T(F)\) are defined differently. There is an important case where we do have symmetry.

Proposition 1.1. Let \(E\) be a reflexive separable Banach space and give \(E'\) the norm topology. Then \(\Omega(E') = \tilde{\Omega}(E')\).

Proof. Comparing the definitions of \(\Omega(E')\) and \(\tilde{\Omega}(E')\), this follows from five facts: (1) If \(p\) is the norm on \(E\), then \(p^0\) is the norm on \(E'\); (2) if \(E\) is separable, so is \(E'\) [8, p. 259]; (3) for a separable Banach space scalar measurability is the same as measurability [1, p. 181]; (4) considering \(E'\) as a Banach space, \(\Omega(E') = \Omega_0(E')\) [9]; (5) if two measurable functions are scalarly a.e. equal, they are a.e. equal [6, p. 21].

Remark. Suppose \(E\) is a reflexive Banach space. Then Proposition 1.1 is true in the following sense. Suppose \(g\) is a member of a class of functions in \(\tilde{\Omega}(E')\). Then by [2, p. 95, Ex. 25] there is a \(g' \equiv g\) such that \(g'\) is a member of a class of functions in \(\tilde{\Omega}(E')\). It is easy to show that the map \(g \rightarrow g'\) induces a well defined, one to one map of \(\tilde{\Omega}(E')\) onto \(\tilde{\Omega}(E')\), giving the result. Using this fact we may drop the hypothesis that \(E\) be separable in Theorem 2.7.

I would like to thank the referee for pointing out the following remark.

Remark. If \(E\) is a Banach space with a separable dual then Proposition 1.1 is still true. For the unit ball in \(E\), being \(\sigma(E'', E')\) dense in the \(\sigma(E'', E')\) compact, metrizable, and hence separable unit ball of \(E''\), contains a countable
set dense in the unit ball of $E''$. Thus by [1, p. 181] any $g \in \Omega(E')$ is norm measurable. By [9, Proposition 3.1] if $g_1 = g_2$ in $\Omega(E')$ then $g_1 = g_2$ a.e. The proof is now similar to the one given in the proposition.

We omit the easy proof of the following proposition.

**Proposition 1.2.** Let $(S(E), T(F))$ be a dual pair of v.f.s.'s. Let $f \in S(E)$ and $B \subseteq T(F)$ be solid. Then

$$\sup \left\{ \left| \int \langle f, g \rangle \, d\pi \right| : g \in B \right\} = \sup \left\{ \int |\langle f, g \rangle| \, d\pi : g \in B \right\}.$$  

It follows that if in the above proposition $S(E)$ is solid, then the polar of $B$ is solid.

**Lemma 1.3.** Let $(S(E), T(F))$ be a dual pair of v.f.s.'s. Then if $T(F)$ is solid, the solid hull of a point $g \in T(F)$ is weakly compact and if $S(E)$ is solid, the solid hull of a point in $S(E)$ is weakly compact.

**Proof.** If $T(F)$ is solid the map $V: L^\infty \rightarrow T(F)$ given by $Vb = bg$ has an adjoint $V^*: S(E) \rightarrow L^1$ given by $V^*f = \langle f, g \rangle$. Thus $V$ is weakly continuous and so the solid hull of $g$, which is the image by $V$ of the weakly compact unit ball of $L^\infty$, is weakly compact. The proof of the other part is similar.

If $(S(E), T(F))$ is a dual pair of v.f.s.'s and $T(F)$ is solid, the normal topology on $S(E)$ is the topology of uniform convergence on the solid hulls of points in $T(F)$. By the above lemma and the Mackey-Arens theorem, the dual of $S(E)$ under the normal topology is $T(F)$.

**Proposition 1.4.** Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s. Then the solid hull of a weakly bounded set in $S(E)$ or $T(F)$ is again weakly bounded.

**Proof.** By the comment above, the bounded sets in the weak and normal topologies on $S(E)$ are the same. By the comment following Proposition 1.2, the normal topology has a base of solid sets. Thus the solid hull of a set bounded in the normal topology is bounded. The proof of the other part is similar.

**Corollary 1.5.** Let $(S(E), T(F))$ be a pair of solid v.f.s.'s. Then the solid hull of a strongly bounded set in $S(E)$ or $T(F)$ is again strongly bounded.

**Proof.** By the proposition, the topology $\beta(S(E), T(F))$ has a base of solid sets from which the first part follows. The proof of the second part is similar.

In several theorems that follow, the statement "let $S(E)$ be a v.f.s. with a topology finer than that induced from $\Omega(E')$" is part of the hypotheses. The following proposition gives a sufficient condition that this hypothesis be satisfied.
PROPOSITION 1.6. For a compact set $K$ and $p \in P$, set
\[ Q(K, p) = \{ g \in \Gamma(F) : p^* g \leq c(K) \}. \]
Let $(S(E), T(F))$ be a dual pair of v.f.s.’s and let $S(E)$ be given a polar topology stronger than the topology of uniform convergence on the sets $Q(K, p)$. Then this topology is stronger than that induced from $\Omega(E)$.

Proof. In the proof of [9 Proposition 6.2] it was shown that
\[ Q(K, p)^0 = \left\{ f \in S(E) : \int_K p \circ f \, d\pi \leq 1 \right\} \]
from which the result follows. 

2. Completeness

The following proposition, due to Garling [5, p. 998] is exactly what we need to establish the completeness of many v.f.s.’s.

PROPOSITION 2.1. Let $G$ be a complete Hausdorff topological vector space with topology defined by a set of seminorms $P$. Let $Q$ be a set of lower semicontinuous extended valued seminorms on $G$. Set
\[ H = \{ g \in G : q(g) < \infty \text{ for all } q \in Q \}. \]
Then $H$ is complete under the topology given by $P \cup Q$.

THEOREM 2.2. Let $(S(E), T(F))$ be a dual pair of solid v.f.s.’s with $S(E) = T(F)^*$. Let $S(E)$ have the topology of uniform convergence on a set $\mathfrak{B}$ of solid sets in $T(F)$ whose union is $T(F)$. Suppose the topology on $S(E)$ is finer than that induced from $\Omega(E)$. Then $S(E)$ is complete.

Proof. By Proposition 1.2, a set of seminorms defining the topology on $S(E)$ is given by $\sup_{\mathfrak{B}} \int | \langle f, g \rangle | \, d\pi$ with $B \in \mathfrak{B}$. This defines a set of (possibly extended valued) seminorms on $\Omega(E)$. We claim that these are lower semicontinuous. Since the supremum of a family of lower semicontinuous functions is lower semicontinuous, it is sufficient to show that $f \to \int | \langle f, g \rangle | \, d\pi$ is lower semicontinuous. We know $Z$ can be expressed as a countable union of compact sets so by the monotone convergence theorem it is sufficient to show that
\[ f \to \int_K | \langle f, g \rangle | \, d\pi \]
is lower semicontinuous for each $K$. Similarly, if we set $g|_K = bg_0$ where $b \in \Omega$ and $p^* g_0 \leq 1$ then it is sufficient to show that $f \to \int_{K'} | \langle f, g \rangle | \, d\pi$ is lower semicontinuous for any $K'$ such that $b|_{K'}$ is bounded. But this is clear and the claim is established. Since $S(E) = T(F)^*$ and since the sets in $\mathfrak{B}$ have union $T(F)$ we have
$$ S(E) = \left\{ f \in \Omega(E) : \sup_{\mathfrak{B}} \int | \langle f, g \rangle | \, d\pi < \infty \text{ for all } B \in \mathfrak{B} \right\}. $$
Thus Proposition 2.1 applies and $S(E)$ is complete.

**Corollary 2.3.** Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s with $S(E) = T(F)^*$. Then $S(E)$ is complete under the strong topology.

**Proof.** By Proposition 1.4, the strong topology is the topology of uniform convergence on all solid weakly bounded sets in $T(F)$. By Proposition 1.6, the topology on $S(E)$ is finer than the topology induced from $\Omega(E)$ since the sets $Q(K, p)$ of Proposition 1.6 are easily seen to be weakly bounded. Thus the theorem applies and $S(E)$ is complete.

**Proposition 2.4.** Let $(S(E), T(E'))$ be a dual pair of solid v.f.s.'s with $S(E) = T(E')^*$ and $T(E') \supseteq \Phi(E')$. Then $S(E)$ is complete under the Mackey topology.

**Proof.** Give $S(E)$ the topology $\xi$ of uniform convergence on the set $\mathcal{G}$ of solid hulls of points in $T(E')$ and the absolutely convex hulls of the solid hulls, $Q'(K, p)$, of the sets $Q(K, p)$ of Proposition 1.6. The sets in $\mathcal{G}$ are solid and absolutely convex. By Proposition 1.6 the topology $\xi$ is stronger than the topology induced from $\Omega(E)$. Thus Theorem 2.2 applies and $S(E)$ is complete under $\xi$. By Lemma 1.3 the solid hulls of points in $T(E')$ are weakly compact. The set $Q'(K, p)$ is weakly relatively compact in $\Phi(E')$, containing the polar of a neighborhood in $\Omega(E)$. Thus $Q'(K, p)$ is also weakly relatively compact in $T(E')$. By the Mackey-Arens theorem, $\xi$ is coarser than the Mackey topology and so $S(E)$ is also complete under the Mackey topology.

**Proposition 2.5.** Let $E$ be a separable Banach space.

(1) If $Z$ is compact, then $\Omega(E)$ is a Banach space and the norm topology on $\Phi(E')$, the dual of $\Omega(E)$, is $\|g\|_\infty$.

(2) If $\pi$ does not have compact support, there is a sequence $K_1 \subseteq K_2 \cdots$ of compact sets such that $\Phi(E')$ is, under the strong topology, the strict inductive limit of the Banach spaces $\Phi_{K_n}(E')$.

**Proof.** (1) $\Omega(E)$ is a Banach space with norm $\int ||f|| \, d\pi$. For any $g \in \Phi(E')$ the function $g$ is $\sigma(E', E)$ measurable and the function $||g||$ is measurable [9, Proposition 3.1]. Thus by [9, Lemma 5.2], for any $a \in L^1$,

$$\int |a||g|| \, d\pi = \sup \left\{ \left| \int \langle f, g \rangle \, d\pi \right| : f \in \Omega(E), \|f\| \leq |a| \right\}.$$

Taking the supremum over $a$ in the unit ball of $L^1$ we have

$$\|\|g\|_\infty = \sup \left\{ \left| \int \langle f, g \rangle \, d\pi \right| : f \in \Omega(E), \int ||f|| \, d\pi \leq 1 \right\}.$$

(2) Let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets such that $K_n \subseteq K_{n+1}^0$ and $\bigcup_{n=1}^\infty K_n = Z$. 
We may assume that $\pi(K_{n+1} - K_n) > 0$ and so $\Phi_n(E')$ is a proper subspace of $\Phi_{K_{n+1}}(E')$. Now $\Phi(E') = \bigcup_{n=1}^{\infty} \Phi_{K_n}(E')$ and so the inductive limit topology on $\Phi(E')$ by the spaces $\Phi_{K_n}(E')$ exists and is a strict inductive limit; it remains to show that this topology is the strong topology. Set $R_n = K_n - K_{n-1}$. Let $\Omega_n(E)$ (respectively $\Phi_n(E')$) be the set of all restrictions of functions in $\Omega(E)$ (respectively $\Phi(E')$) to $R_n$ and give $\Omega_n(E)$ norm $\int_{\Omega_n(E)} |f| d\pi$. Since a function $f : Z \to E$ is measurable iff $f|_{R_n}$ is measurable for all $n$ [1, p. 175], we see that $\Omega(E) = \prod_{n=1}^{\infty} \Omega_n(E)$ algebraically and topologically. Now by arguments similar to those of [9, Theorem 4.1] and (1) of the present theorem (or by [2, p. 47]), $\Omega_n(E)' = \Phi_n(E')$ and the norm on $\Phi_n(E')$ is $\|g\|_\infty$. Thus $\Omega(E)' = \Phi(E')$ is the direct sum of the $\Phi_n(E')$ [11, p. 93] and the direct sum topology on $\Phi(E')$ is the strong topology [11, p. 100]. Thus we must show that the direct sum topology on $\Phi(E')$ coincides with the strict inductive limit topology. Let $(a_n)$ be an arbitrary decreasing sequence of positive numbers. Set

$$N_1(a_n) = \bigcup_{n=1}^{\infty} \{ g \in \Phi(E') : \text{Supp } g \subseteq K_n, \|g\|_\infty \leq a_n \}$$

and

$$N_2(a_n) = \bigcup_{n=1}^{\infty} \{ g \in \Phi(E') : g|_{R_n} = g, \|g\|_\infty \leq a_n \}.$$ 

Let $\langle N_1(a_n) \rangle$ and $\langle N_2(a_n) \rangle$ be the absolutely convex envelopes of $N_1(a_n)$ and $N_2(a_n)$. Then $\langle N_1(a_n) \rangle$ and $\langle N_2(a_n) \rangle$ are arbitrary elements of a base of neighborhoods for the strict inductive limit and direct sum topologies on $\Phi(E')$ respectively. Clearly

$$\langle N_2(a_n) \rangle \subseteq \langle N_1(a_n) \rangle.$$ 

Now we claim that $N_1(n^{-1}a_n) \subseteq \langle N_2(a_n) \rangle$, completing the proof. Let $g \in N_1(n^{-1}a_n)$. Then for some $m$, $\text{Supp } g \subseteq K_m$ and $\|g\|_\infty \leq m^{-1}a_m$. Then

$$g = \sum_{j=1}^{m} n^{-1}(mg|_{K_j}).$$

But

$$\|mg|_{K_j}\|_\infty \leq m(m^{-1}a_m) \leq a_j$$

since $(a_n)$ is decreasing and so $g \in \langle N_2(a_n) \rangle$.}

**Lemma 2.6.** If $E$ is a separable reflexive Banach space, $\Omega(E)$ is weakly sequentially complete.

**Proof.** Let $(f_n)$ be a weakly Cauchy sequence. Then for $g \in \Phi(E')$ and every $b \in L^\infty$ we have $bg \in \Phi(E')$ and so $\int \langle f_n, g \rangle b d\pi$ is a Cauchy sequence of scalars. Thus $(f_n, g)$ is weakly Cauchy in $L^1$ and so has a weak limit [4, p. 92]. Define $\Psi_n : \Phi(E') \to L^1$ by $\Psi_n(g) = \langle f_n, g \rangle$ and $\Psi : \Phi(E') \to L^1$ by $\Psi(g) = = \lim \langle f_n, g \rangle$. We claim that the $\Psi_n$ are continuous when $\Phi(E')$ has the strong dual topology and $L^1$ has the weak topology. Assume that $\pi$ does not have compact support; if it has compact support the proof is simpler. Let $(K_j)$ be a sequence of compact sets such that by Proposition 2.5, $\Phi(E')$ is
the strict inductive limit of the $\Phi_K, (E')$. Since each $\Phi_K, (E')$ is a Banach space, $\Phi, (E')$ is bornological and so sequential continuity of $\Psi, n$ will ensure continuity. Let $g_k \to 0$ in $\Phi, (E')$. Then $(g_n) \subseteq \Phi_K, (E')$ for some $r \in [1, 129]$ and $g_k \to 0$ in $\Phi_K, (E')$. Thus for any $b \in L^\infty$,

$$
\lim_{k \to \infty} \left| \int \Psi_n(g_k) b \, d\pi \right| = \lim_{k \to \infty} \left| \int \langle f_n, g_k \rangle b \, d\pi \right|
$$

$$
\leq \|b\|_\infty \int_{K_r} \|f_n\| \, d\pi \lim_{k \to \infty} \|g_k\|_\infty
$$

$$
= 0
$$

establishing the continuity of $\Psi, n$. Now $\Phi, (E')$, being an inductive limit of barrelled spaces, is barrelled. Thus by the Banach-Steinhaus Theorem, $\Psi$ is continuous.

We now apply [9, Theorem 7.1] (with $E$ and $E'$ switched—see Proposition 1.1) to $\phi(g) = \int \Psi(g) \, d\pi$. For condition (1) of that theorem let $R_j \uparrow R$. Then

\[ \phi(g |_{R_j}) = \int \Psi(g |_{R_j}) \, d\pi \]

\[ = \int \lim_{n \to \infty} \langle f_n, g |_{R_j} \rangle \, d\pi \]

\[ = \lim_{n \to \infty} \int \langle f_n, g |_{R_j} \rangle \, d\pi \]

\[ = \lim_{n \to \infty} \int_{R_j} \langle f_n, g \rangle \, d\pi \]

\[ = \int_{R_j} \lim_{n \to \infty} \langle f_n, g \rangle \, d\pi \]

\[ = \int_{R_j} \Psi(g) \, d\pi. \]

Similarly, $\phi(g |_R) = \int_R \Psi(g) \, d\pi$. By the dominated convergence theorem, $\phi(g |_{R_j}) \to \phi(g |_R)$.

Now let a compact set $K$ be given. Then for some $m, K \subseteq K_m$. Since $\Psi$ is continuous, it is bounded on the unit ball of $\Phi_{K_m}(E')$ from which condition (2) follows. By [9, Theorem 7.1], there is an $f \in \Omega(E)$ such that $\phi(g) = \int \langle f, g \rangle \, d\pi$. Thus

\[ \int \langle f, g \rangle \, d\pi = \int \lim_{n \to \infty} \langle f_n, g \rangle \, d\pi = \lim_{n \to \infty} \int \langle f_n, g \rangle \, d\pi. \]

Thus $f_n \to f$ weakly in $\Omega(E)$ and $\Omega(E)$ is weakly sequentially complete. 

**Theorem 2.7.** Let $E$ be a reflexive separable Banach space. Let
(S(E), T(E')) be a dual pair of solid v.f.s.'s with S(E) = T(E')* and T(E') ⊃ \Phi(E'). Then S(E) is weakly sequentially complete.

Proof. Let \( (f_n) \) be a weakly Cauchy sequence. Then \( (f_n) \) is weakly Cauchy in \( \Omega(E) \) since \( T(E') ⊃ \Phi(E') = \Omega(E)'. \) Thus by Lemma 2.6, \( f_n \to f \) weakly in \( \Omega(E) \). Now let \( g \in T(E') \) be arbitrary. We shall show that

\[
\int \langle f_n, g \rangle \, d\pi \to \int \langle f, g \rangle \, d\pi
\]

thus completing the proof since it will then follow that \( f \in S(E) = T(E')^* \).

As in the proof of Lemma 2.6, \( (f_n, g) \to G \), say, weakly in \( L^1 \). Let \( K_1 \subseteq K_2 \subseteq \cdots \) be a sequence of compact sets such that \( \bigcup K_j = \mathbb{Z} \) and set

\[
R_j = \{ z : \| g(z)\| \leq j \} \cap K_j.
\]

Then \( \bigcup R_j = \mathbb{Z} \). Now \( c(R_j)g \in \Phi(E') \) and so we know from above that

\[
\langle f_n, c(R_j)g \rangle \to \langle f, c(R_j)g \rangle
\]

weakly in \( L^1 \). Thus \( \langle f, g \rangle|_{R_j} = G|_{R_j} \). Therefore \( \langle f, g \rangle = G \) and so

\[
\int \langle f_n, g \rangle \, d\pi \to \int G \, d\pi = \int \langle f, g \rangle \, d\pi.
\]

3. Duals of v.f.s.'s

In this section we study the relationship between the topological and Köthe duals of v.f.s.'s.

Proposition 3.1. Suppose \( S(E) \) is a solid locally convex v.f.s. which is barrelled. Suppose \( S(E)' \supset \Phi(E') \). Then \( S(E)' \supset S(E)^* \).

Proof. Let \( g \in S(E)^* \) be given. Given a compact set \( K \), choose a sequence of relatively compact sets \( R_1 \subseteq R_2 \subseteq \cdots \) such that \( \pi(K - R_j) \leq 1/j \) and \( g|_{R_j} \in \Phi(E') \). This can be done by the definitions of \( \Phi(E') \) and \( \Phi(E') \). By hypothesis, \( g|_{R_j} \in S(E)' \). Also

\[
\int \langle f, g|_{R_j} \rangle \, d\pi \to \int \langle f, g|_{K} \rangle \, d\pi
\]

by the dominated convergence theorem. Thus by the Banach-Steinhaus theorem, \( g|_{K} \in S(E)' \). Similarly, by expressing \( \mathbb{Z} \) as a countable union of compact sets we may prove \( g \in S(E)' \), thus completing the proof.

Proposition 3.2. Suppose \( C \subseteq L^1 \) is weakly compact and suppose \( R_j \uparrow R \). Then

\[
\int_{R_j} |a| \, d\pi \to \int_{R} |a| \, d\pi
\]

uniformly for \( a \in C \).

Proof. Let \( \varepsilon > 0 \) be given. By [4, Théorème 4] choose a compact set \( K \)
such that $\int_{x=\mathbb{R}} |a| \, d\pi < \varepsilon/2$ for $a \in C$. By [4, Théorème 4] again, choose $j_0$ so that if $j \geq j_0$ we have

$$\int_{x=\mathbb{R}} |a| \, d\pi - \int_{x=R_j} |a| \, d\pi < \varepsilon/2$$

for $a \in C$. Thus if $a \in C$ and $j \geq j_0$,

$$\int_{x=\mathbb{R}} |a| \, d\pi - \int_{x=R_j} |a| \, d\pi < \varepsilon$$

and the result follows.

**Proposition 3.3.** Let $(S(E), T(F))$ be a dual pair of solid v.f.s.'s.

1. If $\rho$ is the gauge of the polar of a solid weakly bounded set $B$ in $T(F)$ and $R_j \uparrow R$ then $\rho(f|_{R_j}) \rightarrow \rho(f |_{R})$

2. If $S(E)$ is given the topology of uniform convergence on the solid hulls of the $\sigma(T(F), S(E))$ compact sets and $R_j \uparrow R$ then $f|_{R_j} \rightarrow f|_{R}$.

**Proof.** (1) By Proposition 1.2 and the remark following that proposition,

$$\rho(f) = \text{Sup} \left\{ \int_{x} \langle f, g \rangle \, d\pi : g \in B \right\}.$$

Thus by the monotone convergence theorem

$$\rho(f|_{R_j}) = \text{Sup} \left\{ \int_{R_j} \langle f, g \rangle \, d\pi \right\} \rightarrow \text{Sup} \left\{ \int_{R} \langle f, g \rangle \, d\pi = \rho(f|_{R}) \right\}.$$

(2) Fix $f \in S(E)$. The map $V : T(F) \rightarrow L^1$ given by $Vg = \langle f, g \rangle$ has an adjoint $V^* : L^\infty \rightarrow S(E)$ given by $V^*b = bf$. Thus $V$ is weakly continuous. Let $C$ be a $\sigma(T(F), S(E))$ compact set. By the continuity of $V$, $\langle f, C \rangle$ is a $\sigma(L^1, L^\infty)$ compact set. Thus, if $C'$ is the solid hull of $C$, then $\langle f, C' \rangle$, which is the solid hull of $\langle f, C \rangle$ is, by [4, Théorème 4], $\sigma(L^1, L^\infty)$ compact. Applying Proposition 3.2 we obtain the result.

**Example.** A seminorm on $S(E)$ which is not the gauge of a solid set may fail to have the property of Proposition 3.3 (1) even though the seminorm is induced from $T(F)$. Let $Z$ be the natural numbers with $\pi$ the counting measure. Let $E$ and $E'$ be the real field. Let $S(E)$ and $T(E') = l^\infty$. For every $n$ let $g_n \in l^1$ be defined by

$$g_n(i) = \begin{cases} 1 & \text{if } i = n \\ -1 & \text{if } i = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{g_n\}$. Then $B$ is $\sigma(l^1, l^\infty)$ bounded. Let $\rho$ be the gauge of $B^0$. Let $f \in l^\infty$ be defined by $f(i) = 1$. Then $\rho(f) = 0$ but $\rho(f|_{n, n}) = 1$ for any $i$.

**Proposition 3.4.** Let $E$ be a normed space and $S(E)$ a solid locally convex
v.f.s. For a compact set $K$ define
\[ A(K) = \{ f \in S(E) : \|f\| \leq c(K) \}. \]

(1) Assume $S(E) \supseteq \tilde{\Phi}(E)$ and the topology on $S(E)$ is solid. Let $T(E') \supseteq \tilde{\Phi}(E')$ be a v.f.s. Then the topology on $S(E)$ is a polar topology induced from $T(E')$ if $\rho$ is the gauge of a solid neighborhood and $R_j \uparrow R$ then $\rho(f|_{R_j}) \rightarrow \rho(f|_R)$, $A(K)$ is bounded for every $K$, and $S(E)' \subseteq T(E')$.

(2) We have $S(E)' \subseteq S(E)^*$ if $R_j \uparrow R$ implies $f|_{R_j} \rightarrow f|_R$ and $A(K)$ is bounded for every $K$.

Proof. (1) ($\Rightarrow$) We first show that $A(K)$ is bounded in the topology $\beta(S(E), T(E'))$ and thus in any weaker topology. It follows from Proposition 3.1 that $\Omega^* = \Phi$. It follows from [9, Lemma 5.3] that $\Phi(E) = \Omega(E')^*$. (We shall prove a more general result in [10].) Give $(I)(E)$ the Mackey topology $\tau(\Phi(E), \tilde{\Omega}(E'))$. By Proposition 2.4, $\Phi(E)$ is complete and so in its dual, $\tilde{\Omega}(E')$, the weakly and strongly bounded sets coincide. Now let $B$ be any $\sigma(T(E'), S(E))$ bound set. Then $B$ is $\sigma(\tilde{\Omega}(E'), \Phi(E))$ bounded and so by the above $\beta(\tilde{\Omega}(E'), \Phi(E))$ bounded. Clearly $A(K) \subseteq \Phi(E)$ and is $\sigma(\Phi(E), \tilde{\Omega}(E'))$ bounded. Thus
\[ \sup \left\{ \int \langle f, g \rangle \; d\pi \mid f \in A(K), g \in B \right\} < \infty \]
and so $A(K)$ is $\beta(S(E), T(E'))$ bounded.

The condition on $\rho$ follows from Proposition 3.3. The containment $S(E)' \supseteq T(E')$ is clear.

($\Leftarrow$) Let $B$ be a solid closed absolutely convex neighborhood and $\rho$ its gauge. We shall show that $B$ is closed in $\Omega(E)$. It will then follow that $B$ is $\sigma(\Omega(E), \tilde{\Omega}(E'))$ and so, $\sigma(S(E), T(E'))$ closed. Thus $B^{w^*} = B$ and the result will follow.

To this end, let $(f_n) \subseteq B$ satisfy $f_n \rightarrow f \in \Omega(E)$. Then using [1, p. 131, Théorème 3, 2°] and a diagonal process there is a subsequence, again denoted $(f_n)$ such that $f_n \rightarrow f$ a.e. Let a compact set $K$ and positive integer $k$ be given. Then by Egoroff's theorem [1, p. 175] there is a compact set $K_k \subseteq K$ with $\pi(K - K_k) \leq 1/k$ and $f_n \rightarrow f$ uniformly on $K_k$. We may assume $K_k \subseteq K_{k+1}$. Now
\[ |\rho(f_n|_{K_k}) - \rho(f|_{K_k})| \leq \rho((f - f_n)|_{K_k}) \rightarrow 0 \]
as $n \rightarrow \infty$ since $A(K)$ is bounded. Since $\rho(f_n|_{K_k}) \leq 1$ it follows that $\rho(f|_{K_k}) \leq 1$. But by hypothesis $\rho(f|_{K_k}) \rightarrow \rho(f|_K)$ and so $\rho(f|_K) \leq 1$. By expressing $Z = \bigcup K_i$ where the $K_i$ are compact and $K_i \uparrow Z$ we find that $\rho(f) \leq 1$, i.e., $f \in B$. Thus $B$ is closed in $\Omega(E)$.

(2) ($\Rightarrow$) Proposition 3.3(2) and the Mackey-Arens theorem show that $R_j \uparrow R$ implies $f|_{R_j} \rightarrow f|_R$. The set $A(K)$ is clearly weakly bounded and so bounded.

($\Leftarrow$) This follows immediately from [9, Theorem 7.1].
By combining the hypotheses of Propositions 3.1 and 3.4 (2) one can obtain conditions under which \( S(E)' = S(E)* \). We turn however to the nicer situation in which the topology on \( S(E) \) is a polar topology induced from \( S(E)* \).

**Theorem 3.5.** Let \( E \) be a Banach space. Let \((S(E), T(E'))\) be a dual pair of solid v.f.s.'s with \( S(E)* = T(E') \) and \( S(E) \supseteq \Phi(E) \). Let \( S(E) \) be provided with a polar topology. Then the following are equivalent:

1. \( S(E)' = S(E)* \),
2. If \( R_i \uparrow R \) then \( f|_{R_i} \to f|_R \).

If, in addition, \( Z \) is second countable and \( E \) is separable, the above are equivalent to:

3. \( S(E) \) is separable.

**Proof.** (1) \( \Rightarrow \) (2). This follows immediately from Proposition 3.3 (2) and the Mackey-Arens theorem.

(2) \( \Rightarrow \) (1). Clearly \( S(E)* \subseteq S(E)' \). In the proof of Proposition 3.4 (1) it was shown that \( A(K) \) is bounded and so the result follows from Proposition 3.4 (2).

Now we suppose that \( Z \) is second countable and \( E \) is separable.

(1) \( \Rightarrow \) (3). It is sufficient to show that \( S(E) \) is \( \sigma(S(E), T(E')) \) separable. Since \( \Gamma(E) \) separates points of \( T(E') \) it is weakly dense in \( S(E) \). Let \( X \) be a countable dense set in \( E \) and let \( \mathcal{K} \) be the countable collection of compact sets constructed in [9, Corollary 6.3]. Set

\[
A = \{ f \in \Gamma(E) : f = \sum c(K_j)x_j \text{ with } K_j \in \mathcal{K} \text{ and } x_j \in X \}
\]

and

\[
A' = \{ f \in \Gamma(E) : f = \sum c(R_j)x_j \text{ with } x_j \in X \}.
\]

We shall show that \( A \) is weakly dense in \( A' \) and \( A' \) is weakly dense in \( \Gamma(E) \), finishing this part of the proof since \( A \) is a countable set.

Let \( f = \sum c(R_j)x_j \in A' \) be given and choose, for each \( j \) and \( n \) a \( K_{nj} \in \mathcal{K} \) with \( \pi(R_j - K_{nj}) + \pi(K_{nj} - R_j) < n^{-1} \). This can be done because of the way \( \mathcal{K} \) was constructed and because of the regularity of the measure. Set \( f_n = \sum c(K_{nj})x_j \). Then \( f_n \in A \) and a simple calculation shows \( f_n \to f \) weakly in \( S(E) \).

Now let \( f = \sum c(R_j)x_j \in \Gamma(E) \) be given and for each \( j \), pick a net \( (x_{j,a}) \) such that \( x_{j,a} \in X \) and \( x_{j,a} \to x_j \). We may index each of the nets \( (x_{j,a}) \) with the same index set, a base of neighborhoods in \( E \). Set \( f_a = \sum c(R_j)x_{j,a} \in A' \). The map \( V : S(E) \) given by \( Vx = c(R)x \) has, by [9, Proposition 6.1 (2)], an adjoint \( V^* : T(E') \to E' \) given by \( V^*g = \int_\mathbb{R} g \, d\pi \). Thus \( V \) is weakly continuous and it follows that \( f_a \to f \) weakly in \( S(E) \).

(3) \( \Rightarrow \) (2). Let \( S(E) \) have the topology of uniform convergence on a set of absolutely convex sets \( \mathcal{B} \). We first show that if \( B \in \mathcal{B} \) and if \( (g_j) \) is any sequence in \( B \), then there is a subsequence \( (g_{j_k}) \) such that for any \( f \in S(E) \), \( \lim (f, g_{j_k}) \) exists in the topology \( \sigma(L^1, L^\infty) \). Let \( \{f_n\} \) be dense in \( S(E) \). Since
for a fixed \( n \) the sequence \( \left( \int f_n \cdot g \, d\pi \right) \) is bounded, we may, by a diagonal procedure, pick a subsequence, which we again denote \( \left( g_i \right) \), such that \( \lim \int f_n \cdot g \, d\pi \) exists for each \( n \). Let \( f \in S(E) \) and \( \varepsilon > 0 \) be given. Since

\[
\left\{ g_p - g_q \mid p, q = 1, 2, \ldots \right\} \subseteq 2B,
\]
we can find an \( f_n' \) in \( \{ f_n \} \) so that \( \left| \int \left( f - f_n' \right) \cdot g_p - g_q \, d\pi \right| < \varepsilon \) for all \( p \) and \( q \).

Since \( \lim_{i \to \infty} \int f_n' \cdot g \, d\pi \) exists, we may find an index \( N \) such that if \( p, q \geq N \), then \( \left| \int f_n' \cdot g_p - g_q \, d\pi \right| < \varepsilon \). Combining the two inequalities gives

\[
\int \langle f, g_p - g_q \rangle \, d\pi < 2\varepsilon \quad \text{for} \quad p, q \geq N,
\]
i.e., \( \lim_{j \to \infty} \int \langle f, g \rangle \, d\pi \) exists. Since \( S(E) \) is solid this implies that for any \( a \in L^\infty \), \( \lim_{j \to \infty} \int \langle a(f), g_j \rangle \, d\pi \) exists, i.e., \( \langle f, g_j \rangle \) is weakly Cauchy in \( L^1 \). Since \( L^1 \) is weakly sequentially complete [4, p. 92], \( \lim_{j \to \infty} \langle f, g_j \rangle \) exists in the topology \( \sigma (L^1, L^\infty) \).

With this established we prove (2). Suppose that \( R_n \uparrow R \) but there is an \( f \in S(E) \) such that \( f \mid R_n \) is false. Set \( S_n = R - R_n \). Then there is a \( B \in \mathfrak{B} \), an \( \varepsilon > 0 \), and a sequence \( \{ g_n \} \subseteq B \) such that for all \( n \), \( \left| \int s_n \langle f, g_n \rangle \, d\pi \right| \geq \varepsilon \). By the above we may assume by taking a subsequence, that \( \lim \int f_n \cdot g_n \, d\pi \) exists and so the set \( \{ \langle f, g_n \rangle \} \) is weakly relatively compact in \( L^1 \). But then by Proposition 3.2, \( \lim \int s_n \langle f, g_n \rangle \, d\pi = 0 \), which is a contradiction.

We are now able to show that a conjecture of C&C [3, p. 609] is true.

**Corollary 3.6.** Let \( E \) be a Banach space. Let \( (S(E), T(E')) \) be a dual pair of solid v.f.s. with \( T(E') = S(E)^* \) and \( S(E) \supseteq \hat{\Phi}(E) \). Then the absolutely convex hull of the solid hull of a \( \sigma (T(E'), S(E)) \) compact set is again weakly relatively compact.

**Proof.** Let \( S(E) \) be given the topology \( \xi \) of uniform convergence on the solid hulls of the weakly compact sets in \( T(E') \). By Proposition 3.3 and Theorem 3.5, \( (S(E), \xi) = T(E') \). Now let \( B \) be the solid hull of any weakly compact set of \( T(E') \). Then \( B^0 \) is a \( \xi \)-neighborhood and so \( B^0 \subseteq \hat{\Phi}(E) \), which contains the absolutely convex hull of \( B \), is weakly compact.

**Proposition 3.7.** Let \( E \) be a Banach space. Let \( (S(E), T(E')) \) be a dual pair of solid v.f.s. with \( T(E') = S(E)^* \) and \( S(E) \supseteq \Phi(E) \). Let \( S(E) \) be given a topology of the dual pair and \( T(E') \) the topology of uniform convergence on the precompact sets in \( S(E) \). Then \( T(E') \) is quasicomplete.

**Proof.** Let \( (g_n) \) be a Cauchy net in \( T(E') \) contained in a bounded set \( B \). Let \( G \) be the strong dual of \( S(E) \) and let \( G \) be given the topology of uniform convergence on the precompact sets in \( S(E) \). Let \( \hat{B} \) be the closure of \( B \) in \( G \). Then \( \hat{B} \) is compact in the topology \( \sigma (G, S(E)) \), being contained in the polar of a strong neighborhood. Thus \( \hat{B} \) is compact in the topology of precompact convergence [11, p. 106]. Thus \( g_n \to \phi \in G \). We shall show, using [9, Theorem 7.1], that \( \phi \in T(E') \), completing the proof.
For condition (1) of that theorem let $f \in S(E)$ and $\varepsilon > 0$ be given and suppose that $R_j \uparrow R$. Set $S_j = R - R_j$. Then by Theorem 3.5, we have $f|_{S_j} \to 0$ and so $\{f|_{S_j}\}$ is precompact. Thus there is an $a_0$ such that we have

$$|\langle f|_{S_j}, \phi - g_{a_0}\rangle| \leq \varepsilon/2 \quad \text{for all } j.$$

Now choose $j_0$ so that $|\langle f|_{S_j}, g_{a_0}\rangle| \leq \varepsilon/2$ for all $j \geq j_0$. Then if $j \geq j_0$,

$$|\phi(f|_{S_j})| \leq |\langle f|_{S_j}, \phi - g_{a_0}\rangle| + |\langle f|_{S_j}, g_{a_0}\rangle| \leq \varepsilon.$$

Thus $\phi(f|_{S_j}) \to 0$, i.e., $\phi(f|_{S_j}) \to \phi(f|_R)$. For condition (2) note that by the proof of Proposition 3.4 (1) we have $A(K)$ bounded in the strong topology and so $\phi(A(K))$ is bounded.

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