World's Fastest Derivation of the Lorentz Transformation[1]

Alan Macdonald

Luther College, Decorah, IA 52101, U.S.A.
http://faculty.luther.edu/˜macdonal
macdonal@luther.edu

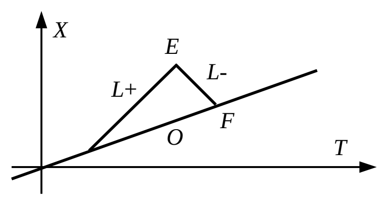
Assume:

(A) The speed of light is the same in all inertial frames. (Take $c = 1$.)

(B) A clock moving with constant velocity v in an inertial frame I runs at a constant rate $\gamma = \gamma(|v|)$ with respect to the synchronized clocks of I which it passes.[2]

Assumption (B) follows from the relativity principle.[3] We do not assume that the Lorentz transformation is linear.

In the figure, E is an arbitrary event plotted in an inertial frame I, $L+$ and $L-$ are the two light worldlines through E, and O is the worldline of the spatial origin of an inertial frame I' moving with velocity v in I. On O,



$$X' = 0,\ X = vT,\ \text{and}\ T = \gamma T'.$$

Thus on O,

$$T + X = \gamma(1 + v)(T' + X') \tag{1}$$
$$T - X = \gamma(1 - v)(T' - X'). \tag{2}$$

Since $c = 1$ in I, an increase in T along $L-$ is accompanied by an equal decrease in X. Thus $T + X$ is the same at E and F. Likewise, since $c = 1$ in I', $T' + X'$ is the same at E and F. Thus Eq. (1), which is true at F, is also true at E. Similar reasoning using $L+$ proves Eq. (2) true at E. Add and subtract Eqs. (1) and (2):

$$T = \gamma(T' + vX') \tag{3}$$
$$X = \gamma(vT' + X'). \tag{4}$$

For $X = 0$ in Eq. (4), $X' = -vT'$; the origin of I has velocity $-v$ in I'.[4] Thus, switching I and I' and using (B), the reasoning for Eq. (1) also gives $T' + X' = \gamma(1 - v)(T + X)$. Substituting this in Eq. (1) gives $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

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[1]Improved from Am. J. Phys. **49** 493 (1981), with a new title.

[2]This does not assume time dilation: as far as (B) is concerned, γ could be identically 1.

[3]Inertial frames are homogeneous and spatially isotropic. Thus $\gamma = \gamma(|v|)$ in a given inertial frame. By the relativity principle, all frames have the same $\gamma(|v|)$.

[4]This *reciprocity principle*, just proved, is often assumed.