Assume: (A) The speed of light is the same in all inertial frames. (Take \(c = 1\).) (B) Inertial frames are homogeneous and spatially isotropic.

Let an inertial clock move with speed \(v\) in an inertial frame \(I\). Let \(\tau = \tau(t)\) be the reading of the clock. By (B), \(d\tau/dt\) is the same for all inertial clocks with speed \(v\) in \(I\). Then by the relativity principle, \(d\tau/dt\) is the same for inertial clocks with speed \(v\) in other inertial frames. Set \(\gamma = \gamma(v) = d\tau/dt\).

In the figure, \(E\) is an arbitrary event in \(I\), \(L^+\) and \(L^-\) are the two light worldlines through \(E\), and \(O\) is the worldline of the spatial origin of an inertial frame \(I'\) moving with velocity \(v\) in \(I\). On \(O\),

\[
X' = 0, \quad X = vT, \quad \text{and} \quad T = \gamma T'.
\]

Thus on \(O\),

\[
\begin{align*}
T + X &= \gamma(1 + v)(T' + X') \quad (1) \\
T - X &= \gamma(1 - v)(T' - X'). \quad (2)
\end{align*}
\]

Since \(c = 1\) in \(I\), an increase in \(T\) along \(L^-\) is accompanied by an equal decrease in \(X\). Thus \(T + X\) is the same at \(E\) and \(F\). Likewise, since \(c = 1\) in \(I'\), \(T' + X'\) is the same at \(E\) and \(F\). Thus Eq. (1), which is true at \(F\), is also true at \(E\). Similar reasoning using \(L^+\) proves Eq. (2) true at \(E\). Add and subtract Eqs. (1) and (2):

\[
\begin{align*}
T &= \gamma(T' + vX') \quad (3) \\
X &= \gamma(vT' + X'). \quad (4)
\end{align*}
\]

For \(X = 0\) in Eq. (4), \(X' = -vT'\); the origin of \(I\) has velocity \(-v\) in \(I'\). Thus, switching \(I\) and \(I'\) and using (B), the reasoning for Eq. (1) also gives \(T' + X' = \gamma(1 - v)(T + X)\). Substituting this in Eq. (1) gives \(\gamma = (1 - v^2)^{-\frac{1}{2}}\).

Equations (3) and (4), with \(\gamma = (1 - v^2)^{-\frac{1}{2}}\) are the Lorentz transformations at the arbitrary event \(E\).

---

2macdonal@luther.edu, http://faculty.luther.edu/~macdonal.
3We do not assume that the Lorentz transformation is linear.
4This does not assume time dilation: as far as (B) is concerned, \(\gamma\) could be identically 1.
5This reciprocity principle, just proved, is often assumed.