Sturm-Liouville Theory Via Nonstandard Analysis

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1. Introduction. It is the purpose of this paper to use the tools of nonstandard analysis to solve Sturm-Liouville systems of differential equations. A complete statement of what we shall prove is given in the main theorem below. A more general equation and more general boundary conditions than in (1) could have been handled but would have made the exposition technically more difficult.

Not much of nonstandard analysis is needed; the paper by Luxemburg ([6]) is more than enough and is easily accessable to nonlogicians. Other references are ([5]) and ([7]). If x and y are nonstandard real numbers $x =_1 y$ will mean that $x - y$ is infinitesimal. We shall use ${}^0x$ to denote the standard part of x. Recall that a standard sequence $x_n \to x$ if and only if $x_n =_1 x$ for all infinite n.

Two standard approaches to the theory of Sturm-Liouville systems can be found in ([2]) and ([1]). The present approach seems considerably simpler in conception and execution.

Occasionally in the paper we shall use standard results of analysis (*e.g.*, Bessel's inequality and the uniqueness of solutions of second order linear differential equations with initial conditions) when standard proofs are simple and a nonstandard proof would not be particularly illuminating.

Main Theorem. *Consider the differential equation*

$$Y''(x) - Q(x)Y(x) + \lambda Y(x) = F(x),\ Y(0) = Y(1) = 0 \tag{1}$$

and the associated homogeneous equation

$$Y''(x) - Q(x)Y(x) + \lambda Y(x) = 0,\ Y(0) = Y(1) = 0 \tag{2}$$

where Q and F are continuous and Q is non-negative. Let $Q_1 \leqq Q(x) \leqq Q_2$ on [0, 1] *Then:*

(*a*) *The equation* (2) *has a non-zero solution only for a countably infinite set of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$, where $j^2\pi^2 + Q_1 \leqq \lambda_j \leqq j^2\pi^2 + Q_2$. For each eigenvalue λ_j there is a unique normalized solution $Y_j(x)$. If $i \neq j$, then $(Y_i, Y_j) = 0$. The functions Y_j are uniformly bounded. For a function $G(x)$ with two con-*

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tinuous derivatives and satisfying $G(0) = G(1) = 0$ *we have*

$$G(x) = \sum_{j=1}^{\infty} (G, Y_j) Y_j(x),$$

where the series converges uniformly and absolutely.

(*b*) *If for every* j, $\lambda \neq \lambda_j$, *the equation* (1) *has a unique solution*

$$Y(x) = \sum_{j=1}^{\infty} \frac{(F, Y_j)}{\lambda - \lambda_j} Y_j(x)$$

where the series and its derivative converge uniformly and absolutely. If $\lambda = \lambda_n$, *the equation* (1) *has a solution only if* $(F, Y_n) = 0$. *If* $(F, Y_n) = 0$, *then*

$$Y(x) = \sum_{j \neq n} \frac{(F, Y_j)}{\lambda - \lambda_j} Y_j(x)$$

is a solution and any other solution differs from this one by a constant multiple of Y_n. *The series and its derivative converge uniformly and absolutely.*

2. The associated difference equation and its solution. Let N be an arbitrary but fixed infinite natural number. Set $q(k) = Q(k/N)$, $f(k) = F(k/N)$, and $y(k) = Y(k/N)$ for $k = 0, 1, \cdots, N$. To motivate the difference equation which will be used to solve (1) and (2) we need to observe that if h is a function defined on $[0, 1]$ and $h''(x)$ exists and is continuous then

$$h''(x) = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - 2h(x) + h(x - \Delta x)}{(\Delta x)^2} \tag{3}$$

uniformly in x. (Proof: Expand h in a Taylor series with two terms plus a remainder and use the uniform continuity of h''.) For any sequence $a(k)$ set $\Delta a(k) = a(k + 1) - a(k)$ and $\Delta^2 a(k) = a(k + 1) - 2a(k) + a(k - 1)$. (Note that $\Delta^2 a(k) = \Delta(\Delta a(k - 1))$.) In view of (3) we have for $x =_1 k/N$, $Y''(x) = Y''(k/N) =_1 N^2 \Delta^2 y(k)$. Thus we shall study the difference equations

$$N^2 \Delta^2 y(k) - q(k)y(k) + \tau y(k) = f(k),\ y(0) = y(N) = 0 \tag{4}$$

and

$$N^2 \Delta^2 y(k) - q(k)y(k) + \tau y(k) = 0,\ y(0) = y(N) = 0. \tag{5}$$

It turns out that (4) and (5) can easily be solved. Then using the solutions of (4) and (5) we shall construct solutions of (1) and (2).

First of all note that there cannot be linearly independent solutions of (5) with the same τ. For any solution is entirely determined by recursion by $y(1)$ and $y(0) = 0$. Thus if y_1 and y_2 are solutions and $y_1(1) = cy_2(1)$, then $y_1 = cy_2$.

Consider the quantities $y(1), y(2), \cdots, y(N - 1)$ as components of an $N - 1$ dimensional vector $\mathbf{y}$. Define an $(N - 1) \times (N - 1)$ matrix $A = (a_{ij})$ by $a_{ii} = 2N^2 + q(i)$, $a_{i,i+1} = a_{i,i-1} = -N^2$, and $A_{ij} = 0$ otherwise. Then (5) is equivalent to

$$A\mathbf{y} = \tau\mathbf{y},\ y(0) = y(N) = 0.$$

Since A is a symmetric matrix, there are exactly $N - 1$ orthogonal solutions $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{N-1}$ and $N - 1$ eigenvalues $\tau_1 < \tau_2 \cdots < \tau_{N-1}$. (Strict inequality because of the comments above.) Translating this in terms of (5) there are $N - 1$ orthogonal solutions $y_1, y_2, \cdots, y_{N-1}$ and $N - 1$ eigenvalues $\tau_1 < \tau_2 \cdots < \tau_{N-1}$. It will be convenient to take

$$||y_i||^2 = (y_i, y_i) = N. \tag{6}$$

We now solve the non-homogeneous equation (4). If $\tau \neq \tau_j$ for all j, then a solution of (4) is

$$y(k) = \frac{1}{N}\sum_{j=1}^{N-1}(y_j, f)\frac{y_j(k)}{\tau - \tau_j} \tag{7}$$

since

$$\begin{aligned} N^2\Delta^2 y - qy + \tau y &= \frac{N^2}{N}\sum_{j=1}^{N-1}(y_j, f)\frac{\Delta^2 y_j(k)}{\tau - \tau_j} - (q + \tau)\frac{1}{N}\sum_{j=1}^{N-1}(y_j, f)\frac{y_j(k)}{\tau - \tau_j} \\ &= \frac{1}{N}\sum_{j=1}^{N-1}(N^2\Delta^2 y_j - qy_j + \tau_j y_j - \tau_j y_j + \tau y_j)\frac{(y_j, f)}{\tau - \tau_j} \\ &= \frac{1}{N}\sum (y_j, f)y_j = f \end{aligned}$$

since the last sum is just the Fourier expansion of f in view of (6). Similarly, if $\tau = \tau_n$ and $(f, y_n) = 0$, then a solution of (4) is

$$y(k) = \frac{1}{N}\sum_{j\neq n}(y_j, f)\frac{y_j(k)}{\tau - \tau_j}. \tag{8}$$

3. A special difference equation. In order to obtain more information about the y_j and τ_j it will be necessary to obtain explicit solutions of

$$N^2\Delta^2 s(k) + \sigma s(k) = 0, \qquad s(0) = 0, \qquad s(1) = \frac{\sigma^{1/2}}{N} \tag{9}$$

for $0 < \sigma < 4N^2$. Standard methods for solving homogeneous linear difference equations with constant coefficients ([4], p. 26) provide the unique solution

$$s_\sigma(k) = c \operatorname{Sin} \hat{\sigma}\frac{k}{N} \tag{10}$$

where $c = c(\sigma)$ is chosen so that $s_\sigma(1) = (\sigma)^{1/2}/N$ and

$$\hat{\sigma} = N\,\operatorname{Tan}^{-1}\frac{\sigma^{1/2}(4N^2 - \sigma)^{1/2}}{2N^2 - \sigma} \tag{11}$$

where Tan^{-1} takes values in the 1st and 2nd quadrants. The solution may be verified directly by plugging (10) into (9), using $\mathrm{Sin}\,(x + y) = \mathrm{Sin}\, x\, \mathrm{Cos} y + \mathrm{Cos} x\, \mathrm{Sin} y$, and then using the identity $\mathrm{Cos}^{-1} x = \mathrm{Tan}^{-1} (1 - x^2)^{1/2}/x$. The qualitative behavior of $\hat{\sigma}$ is deduced by noting that as $\hat{\sigma}$ increases continuously from 0 to $2N^2$, $\sigma^{1/2}(4N^2 - \sigma)^{1/2}$ increases and so $\hat{\sigma}$ increases continuously from 0 to $N\pi/2$. Since $\mathrm{Tan}^{-1}(-x) = \pi - \mathrm{Tan}^{-1}x$ we see that

$$4N^2 - \rho = N\pi - \hat{\rho}. \tag{12}$$

and so as σ increases from $2N^2$ to $4N^2$, $\hat{\sigma}$ increases from $N\pi/2$ to $N\pi$. Thus if s is also to satisfy

$$s_\sigma(N) = 0 \tag{13}$$

we may pick eigenvalues σ_i so that

$$\hat{\sigma}_i = j\pi \qquad j = 1, 2, \cdots, N - 1. \tag{14}$$

We must now estimate $\hat{\sigma}$ and σ_i. I claim that

$$\frac{3}{4}\sigma^{1/2} \leqq \hat{\sigma} \leqq 5\sigma^{1/2}, \tag{15}$$

$$\frac{1}{25j^2} \leqq \frac{1}{\sigma_i} \leqq \frac{3}{j^2}, \tag{16}$$

$$\hat{\sigma} = {}_{|}(\sigma)^{1/2} \tag{17}$$

for σ finite, and

$$\sigma_i = {}_{|}\, j^2\pi^2 \text{ and } \sigma_{N-i} = {}_{|}\, 4N^2 - j^2\pi^2 \tag{18}$$

and j finite.

For $N^2/2 \leqq \sigma \leqq 4N^2$, we have $N\pi/2 \leqq \hat{\sigma} \leqq N\pi$ and so

$$\sigma^{1/2} \leqq 2N \leqq \frac{4}{\pi}\hat{\sigma} \leqq 4N \leqq (32\sigma)^{1/2}$$

verifying (15) in this case. Using $d/d\theta\ \mathrm{Tan}^{-1}\theta = (1 + \theta^2)^{-1}$ and expanding the geometric series it is easy to obtain

$$\theta - \frac{\theta^3}{3} \leqq \mathrm{Tan}^{-1}\,\theta \leqq \theta \qquad 0 \leqq \theta \leqq 1. \tag{19}$$

We apply these inequalities in (11), setting $\sigma = \alpha N^2$, $0 \leqq \alpha \leqq 1/2$. (Note that the inequalities (19) apply for $\sigma \leqq N^2/2$.) Now

$$\hat{\sigma} = N\,\mathrm{Tan}^{-1}\frac{\sigma^{1/2}(4N^2 - \sigma)^{1/2}}{2N^2 - \sigma} \leqq N\frac{\sigma^{1/2}(4N^2 - \sigma)^{1/2}}{2N^2 - \sigma} \tag{20}$$

$$\leqq N\frac{\sigma^{1/2}(4N^2)^{1/2}}{(2 - \alpha)N^2} = \frac{2\sigma^{1/2}}{2 - \alpha}$$

Using the other inequality in (19) we obtain

$$\hat{\sigma} \geqq N\,\mathrm{Tan}^{-1}\frac{\sigma^{1/2}(4 - \alpha)^{1/2}N}{2N^2} \geqq \sigma^{1/2}\left[\frac{(4 - \alpha)^{1/2}}{2} - \frac{\alpha(4 - \alpha)^{3/2}}{24}\right]. \tag{21}$$

Since $0 \leqq \alpha \leqq 1/2$, simple estimates from (20) and (21) show (15) is valid for $0 \leqq \sigma \leqq N^2/2$ also. Now $\hat{\sigma}_j = j\pi$ so (16) follows from (15). If σ is finite in (20) and (21) then $\alpha =_1 0$ and this yields (17). If j is finite, then so is σ_j by (16). Then (17), (14), and (12) prove (18).

Now we shall estimate c and $||s_\sigma||$. We shall be interested in the range

$$3/4 \leqq \sigma \leqq 4N^2 - 3/4. \tag{22}$$

From (9), (10), (12), and (17), we have

$$c = \frac{\sigma^{1/2}}{N \operatorname{Sin} \hat{\sigma}/N} \leqq \frac{\sigma^{1/2}}{N \operatorname{Sin} 3/4N} \leqq \frac{\sigma^{1/2}}{3/4} < 2\sigma^{1/2} \tag{23}$$

and for σ finite,

$$c =_1 \sigma^{1/2}/\hat{\sigma} =_1 1. \tag{24}$$

By ([3]), p. 23),

$$\sum_{k=0}^{N} \operatorname{Sin}^2 bk = \left(k/2 - \frac{\operatorname{Sin} b(2k-1)}{4 \operatorname{Sin} b}\right)\Bigg|_0^{N+1} \quad (\operatorname{Sin} b \neq 0).$$

Thus

$$\begin{aligned} ||s_\sigma||^2 &= c^2 N\left(1/2 + 1/4N - \frac{\operatorname{Sin}(2N+1)\hat{\sigma}/N}{4N \operatorname{Sin} \delta/N}\right) \\ &= c^2 N\beta, \qquad 1/7 \leqq \beta \leqq 1, \end{aligned} \tag{25}$$

where the inequality for β follows from an estimate of $N \operatorname{Sin} \hat{\sigma}/N$ similar to that in (23).

4. Comparison of eigenvalues. For any $y(n)$ with $y(0) = y(N) = 0$,

$$(Q_1 y - \Delta^2 y, y) \leqq (qy - \Delta^2 y, y) \leqq (Q_2 y - \Delta^2 y, y).$$

By the (finite dimensional) mini-max principle ([3], p. 181) the eigenvalues of $Q_2 y - \Delta^2 y$ dominate those of $qy - \Delta^2 y$ which in turn dominate those of $Q_1 y - \Delta^2 y$, *i.e.*,

$$\sigma_j + Q_1 \leqq \tau_j \leqq \sigma_j + Q_2 . \tag{26}$$

5. Estimates of y_j. As before, let τ_j and y_j be the j^{th} eigenvalue and eigensolution of (5) and let s_σ be the solution of (9) with $\sigma = \tau_j$. Suppose σ satisfies (22). Set

$$z(k) = y_j(k) - \frac{1}{N\tau_j^{1/2}} \sum_{i=0}^{k} q(i) y_j(i) s_\sigma(k - i).$$

A quick calculation using $s_\sigma(-1) = s_\sigma(1) = \tau_j^{1/2}/N$ ((10) and (9)) shows that $N^2\Delta^2 z + \tau_j z = 0$, $z(0) = 0$. Since solutions of (9) are unique, $z = as_\sigma$ for some constant $a = a(\sigma)$. Thus

$$(27) \qquad y_j(k) = as_\sigma(k) + \frac{1}{N\tau_j^{1/2}} \sum_{i=0}^{k} q(i)y_j(i)s_\sigma(k-i).$$

Now

$$(28) \qquad \frac{1}{N\tau_j^{1/2}} \left| \sum_{i=0}^{k} q(i)y_j(i)s_\sigma(k-i) \right| \leqq \frac{c}{N\tau_j^{1/2}} \|y_j\| \, \|q\| \leqq 2Q_2$$

by (6) and (23). From (27), we have $|asc_\sigma(k)| \leqq |cy_j(k)| + 2cQ_2$. Thus, $a^2c^2s_\sigma^2(k) \leqq c^2y_j^2(k) + 4c^2Q_2|y_j(k)| + 4c^2Q_2^2$. Summing this and using the inequality

$$\sum_{k=1}^{N} 1\cdot|y_j(k)| \leqq N^{1/2}\,\|y_j\| = N$$

we obtain $a^2c^2\|s_\sigma\|^2 \leqq c^2N + 4c^2Q_2N + 4c^2Q_2^2N = c^2N(1 + 4Q_2 + 4Q_2^2)$. Using (25) then yields

$$(29) \qquad a^2c^2 \leqq 7(1 + 4Q_2 + 4Q_2^2).$$

Finally, using (27) and (28) gives

$$(30) \qquad |y_j(k)| \leqq ac + 2Q_2 \leqq M, \qquad M \text{ finite, standard.}$$

The subscripted M's below are finite constants. Applying Δ to both sides of (27), using (10) and the inequality

$$|\operatorname{Sin} x - \operatorname{Sin} y| = \left| \int_y^x \operatorname{Cos} t \, dt \right| \leqq |x - y|.$$

and then using (29) and (23) gives

$$|\Delta y_j(k)| \leqq ac\hat{\sigma}/N + Q_2Mc\hat{\sigma}/\sigma^{1/2}N \leqq M_1\,\hat{\sigma}/N.$$

For $j \leqq j_0$, j_0 finite, $\sigma_j + Q_2 \leqq 2\sigma_j$ by (18). Thus by (15), (26), and (16),

$$\hat{\sigma} = \hat{\tau}_j \leqq 5\tau_j^{1/2} \leqq 5(\sigma_j + Q_2)^{1/2} \leqq 5(2\sigma_j)^{1/2} \leqq M_2 j$$

By making M_2 larger if necessary this holds for $j < j_0$ also. Thus

$$(31) \qquad |\Delta y_j(k)| \leqq M_3 j/N.$$

The estimates (30) and (31) hold so far only under the assumption (22). The inequality $3/4 \leqq \tau_j$ is always valid by (26) and (18), but $\tau_j \leqq 4N^2 - 3/4$ may be violated (by (26) and (18)) for j close to $N - 1$. However, for these values of j we consider y_j as a solution of

$$N^2\Delta^2 y - (q - Q_2)y + (\tau - Q_2)y = 0.$$

The eigenvalues, $\tau_j - Q_2$, of this equation are smaller than $4N^2 - 3/4$. Repeating the arguments of this section with $s_{\sigma - Q_2}$ instead of s_σ shows that (30) and (31) are valid for all j.

6. Construction of the solution. In this section j will be finite. Using (31) we obtain

$$|y_j(k) - y_j(l)| = \left| \sum_{i=l}^{k-1} \Delta y_j(i) \right| \leqq |k - l| \, M_3 j/N. \tag{32}$$

Thus if $k/N =_1 l/N$, then $y_j(k) =_1 y_j(l)$. This allows us to define a standard function $Y_j(x)$ by setting $Y_j(x) = {}^0y_j(k)$ for any value of k such that $x =_1 k/N$. This Y_j will be the solution of the differential equation. Note that Y_j is continuous by (32).

In order to proceed we need the following lemma.

Lemma. *If $h(t)$ is a standard continuous function, $x =_1 l/N \leqq 1$, and $H(k)$, $k = 1, 2, \cdots, l$ is an internal sequence satisfying $h(k/N) =_1 H(k)$ for all k, then*

$$\int_0^x h(t)\, dt =_1 1/N \sum_{k=0}^{l} H(k).$$

Proof. Let S be the largest of the finite (in the nonstandard sense) set of numbers $|h(k/N) - H(k)|$. Then $S =_1 0$. Now

$$\int_0^x h(t)\, dt = \lim_{n\to\infty} 1/N \sum_{k=0}^{[xN]} h(k/N) =_1 1/N \sum_{k=0}^{l} h(k/N).$$

But

$$\left| 1/N \sum_{k=0}^{l} h(k/N) - 1/N \sum_{k=0}^{l} H(k) \right| \leqq S =_1 0.$$

This completes the proof of the lemma.

Using the lemma, (27), (10), and (24) we find, setting $\lambda_j = {}^0\tau_j$,

$$Y_j(x) = {}^0a \operatorname{Sin} \lambda_j^{1/2} x - 1/\lambda_j^{1/2} \int_0^x Q(t) Y_j(t) \operatorname{Sin} \lambda_j^{1/2}(x - t)\, dt. \tag{33}$$

(The constant a is finite by (29).) Differentiating twice shows that Y_j is a solution of (2). Several parts of (*a*) of the main theorem are now accessable. The Y_j are orthogonal since the y_j are orthogonal and so by the lemma (if $i \neq j$),

$$0 = 1/N \sum_{k=0}^{N} y_i(k) y_j(k) =_1 \int_0^1 Y_i(t) Y_j(t)\, dt.$$

Also by the lemma and (6)

$$\int_0^1 Y_j^2(t)\, dt =_1 1/N \sum_{k=0}^{N} y_j^2(k) = 1$$

and so the y_j are normalized.

The eigenvalues λ_j are distinct since any solution Y of (2) is determined by $Y(0)$ and $Y'(0)$ ([1], p. 31). Thus if Y and Z are two solutions with the same λ and $Y'(0) = cZ'(0)$, then $Y = cZ$. The inequalities on the λ_j follows from (26), (14), and (18).

7. Expansion of C^2 functions. Let G be a function with two continuous derivatives and $G(0) = G(1) = 0$ and set $g(k) = G(k/N)$, $k = 0, 1, \cdots, N$. We shall need the formula for summation by parts,

$$\sum_{k=n}^{N} u(k)\ \Delta v(k) = u(k)v(k)\Big|_{n}^{N+1} - \sum_{k=n}^{N} v(k+1)\ \Delta u(k)$$

([4], p. 17). Let $|G''(x)| \leqq P$. For $0 \leqq k \leqq N$ set $x = {}^{\circ}(k/N)$. Using the uniformity of the limit in (3), we have $|N^2\Delta^2 g(k)| =_{\mid} |G''(k/N)| =_{\mid} |G''(x)| \leqq P$. Two summations by parts give

$$(34) \qquad \begin{aligned} \left|\sum_{k=0}^{N} \Delta^2 y_i(k) g(k)\right| &= \left|\sum_{k=0}^{N-1} \Delta(\Delta y_i(k-1)) g(k)\right| \\ &= \left|\sum_{k=0}^{N-1} \Delta y_i(k)\ \Delta g(k)\right| \\ &= \left|\sum_{k=0}^{N-1} y_i(k+1)\ \Delta(\Delta g(k))\right| \\ &= \left|\sum_{k=0}^{N} y_i(k)\ \Delta^2 g(k)\right| \\ &\leqq MP/N \end{aligned}$$

by (30). Using the Fourier expansion of g and the lemma we have for finite n_0

$$\begin{aligned} G(x) &=_{\mid} g(k) = 1/N \sum_{j=1}^{N} (y_j\ , g) y_j(k) \\ &=_{\mid} \sum_{j=1}^{n_0-1} \left(\int_0^1 Y_j(t)G(t)\,dt\right) Y_j(x) + 1/N \sum_{j=n_0}^{N} (y_j\ , g) y_j(k). \end{aligned}$$

Thus to show the uniform convergence of the Fourier series of G it is sufficient to show that the last sum above is arbitrarily small, uniformly in k, for n_0 large. Let $|G(x)| \leqq G$. Then using (30), (34), the Cauchy-Schwartz inequality, (26), and (16) we have

$$(35) \qquad \begin{aligned} 1/N \sum_{j=n_0}^{N} |(y_j\ , g) y_j(k)| &\leqq M/N \sum_{j=n_0}^{N} |(y_j\ , g)| \\ &= M/N \sum_{j=n_0}^{N} 1/\tau_j\ |(qy_j - N^2\ \Delta^2 y_j\ , g)| \\ &\leqq M/N \sum_{j=n_0}^{N} 1/\tau_j (Q_2 N G_2 + NMP) \\ &= M' \sum_{j=n_0}^{N} 1/\tau_j \leqq M' \sum_{j=n_0}^{N} 2/j^2 \\ &=_{\mid} M' \sum_{j=n_0}^{\infty} 2/j^2 \to 0 \end{aligned}$$

as $n_0 \to \infty$. This proves the uniform convergence of the series. The absolute convergence is proved by estimating $|(Y_i\ , G)Y(x)|$ with techniques similar to those in (35).

If Y were any other normalized solution of (1) besides the Y_i it would be orthogonal to each Y_i ([1], p. 279) and so would be zero by the above proved expansion. Thus the Y_i are the only solutions. This completes the proof of (*a*) of the main theorem.

8. Solutions of the non-homogeneous equation. If for every j, $\lambda \neq \lambda_i$, set

$$Y(x) = \sum_{j=1}^{\infty} \frac{(Y_i\ , F)}{\lambda - \lambda_i} Y_i(x). \tag{36}$$

Then formally

$$Y'(x) = \sum_{j=1}^{\infty} \frac{(Y_i\ , F)}{\lambda - \lambda_i}\ Y_i'(x). \tag{37}$$

Using the Cauchy-Schwartz and Bessel inequalities as well as (31) and noting that if $x =_1 k/N$, then $Y_i'(x) =_1 N\Delta y_i(k)$ we have

$$\left|\sum_{j=1}^{\infty} \frac{(Y_i\ , F)}{\lambda - \lambda_i}\ Y_i'(x)\right| \leqq \left(\sum_{j=1}^{\infty} (Y_i\ , F)^2\right)^{1/2} \left(\sum_{j=1}^{\infty} \left(\frac{Y_i'(x)}{\lambda - \lambda_i}\right)^2\right)^{1/2} \tag{38}$$
$$\leqq ||F|| \left(\sum_{j=1}^{\infty} \frac{M_3^{\ 2} j^2}{(\lambda - \lambda_i)^2}\right)^{1/2}.$$

The last series converges by (16) and so the series (37) converges uniformly and absolutely. The proof of the uniform and absolute convergence of (36) is easier, using (30). Thus Y' is represented by (37).

Let a standard $\epsilon > 0$ be given. Choose n_0 so that for all x, $n \geqq n_0$,

$$\left|Y'(x) - \sum_{j=1}^{n} \frac{(Y_i\ , F)}{\lambda - \lambda_i}\ Y_i'(x)\right| < \epsilon. \tag{39}$$

Using the lemma we find

$$\sum_{j=1}^{n} \frac{(Y_i\ , F)}{\lambda - \lambda_i}\ Y_i'(x) =_1 \sum_{j=1}^{n} \frac{(y_i\ , f)}{\lambda - \tau_i}\ \Delta y_i(k). \tag{40}$$

Using techniques similar to those in (35) we obtain

$$\left|\sum_{j=1}^{N-1} \frac{(y_i\ , f)}{\lambda - \tau_i}\ \Delta y_i(k) - \sum_{j=1}^{n} \frac{(y_i\ , f)}{\lambda - \tau_i}\ \Delta y_i(k)\right| < \epsilon \tag{41}$$

for $n > n_0$ (it may be necessary to increase n_0). Let y be the solution of the non-homogeneous difference equation as in (7). Then

$$N\ \Delta y(k) = \sum_{j=1}^{N-1} \frac{(y_i\ , f)}{\lambda - \tau_i}\ \Delta y_i(k). \tag{42}$$

On the other hand, summing the identity $N^2\Delta^2 y(k) = f(k) + q(k) - \lambda y(k)$

and using the lemma gives

$$N\,\Delta y(k) = \frac{1}{N}\sum_{i=0}^{k} (f(i) + q(i)y(i) - \lambda y(i)) + Ny(1) \tag{43}$$
$$= {}_{\text{I}}\int_0^x (F(t) + Q(t)Y(t) - \lambda Y(t))\,dt + Y'(0).$$

It follows from (39) — (43) that

$$Y'(x) = \int_0^x (F(t) + Q(t)Y(t) - \lambda Y(t))\,at + Y'(0).$$

Differentiating this shows that Y is a solution of (1). There can be no other solution since the difference of two solutions is a solution to (2) and is thus, as we have shown, zero.

If $Y(x)$ is a solution of (1) with $\lambda = \lambda_n$, then multiplying both sides of (1) by Y_n and integrating gives

$$(Y'' - QY + \lambda_n Y, Y_n) = (F, Y_n).$$

Two intergrations by parts show that $(Y'', Y_n) = (Y, Y_n'')$ so we have

$$0 = (Y_n'' - QY_n + \lambda_n Y_n, Y) = (F, Y_n).$$

If $(F, Y_n) = 0$ then

$$\sum_{i\neq n} \frac{(F, Y_i)}{\lambda - \lambda_i} Y_i$$

is a solution satisfying everything claimed. The proof is similar to that above, using (8).

This completes the proof of the main theorem.

Remark. The only essential use of sections 3, 4, and 5 is to produce (30), (31), and an estimate of the form $1/\tau_i \leqq C/j^2$ for some finite constant C. (The use of $s_\sigma(k)$ in (33) can be eliminated). It would be desirable to obtain these facts more simply, perhaps using theorems about eigenvectors and eigenvalues of symmetric matrices with special properties, since this would eliminate the least intuitive and most technically involved parts of this paper.

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