Linear and Geometric Algebra

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Geometry without algebra is dumb! - Algebra without geometry is blind!

- David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.

- Allan McRobie and Joan Lasenby

To David Hestenes,

founder, chief theoretician, and most forceful advocate
for modern geometric algebra and calculus,
and inspiration for this book.

To my Grandchildren,

Aida, Pablo, Miles, Graham.

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Preface

Linear algebra is part of the standard undergraduate mathematics curriculum because it is of central importance in pure and applied mathematics. It was not always so. The wide acceptance of vector methods did not occur until early in the twentieth century. The pioneers were two physicists: the Englishman Oliver Heaviside and the American Josiah Willard Gibbs, beginning in the late 1870's. Linear algebra allows easy algebraic manipulation of vectors. But it is not the latest word on the algebraic manipulation of geometric objects.

Geometric algebra is an extension of linear algebra pioneered by the American physicist David Hestenes in the 1960's. Geometric algebra and its extension to geometric calculus unify, simplify, and generalize vast areas of mathematics, including linear algebra, vector calculus, exterior algebra and calculus, tensor algebra and calculus, quaternions, real analysis, complex analysis, and euclidean, noneuclidean, and projective geometries. They provide a common mathematical language for many areas of physics (classical and quantum mechanics, electrodynamics, special and general relativity), computer science (graphics, robotics, computer vision), engineering, and other fields.[1]

Just as linear algebra algebraically manipulates one dimensional objects (represented by vectors) in a coordinate-free manner, geometric algebra algebraically manipulates higher dimensional objects – lines, planes, ... (represented by multivectors) in a coordinate-free manner. Even within linear algebra, many topics are improved by using geometric algebra.

Geometric algebra subsumes, unifies, simplifies, and generalizes the vector, complex, quaternion, exterior (Grassmann), and tensor algebras.

I believe that it is past time to incorporate some geometric algebra in the introductory linear algebra course. This book provides a text for such a course. Single variable calculus is not a prerequisite. But for most students a mathematical maturity equivalent to that gained in such a course probably is.

[1]Several advanced geometric algebra books have appeared since 2007: *Understanding Geometric Algebra: Hamilton, Grassmann, and Clifford for Computer Vision and Graphics* (2015); *Understanding Geometric Algebra for Electromagnetic Theory* (2011); *Geometric Algebra with Applications in Engineering (2009)*; *Invariant Algebras and Geometric Reasoning (2008)*; *Geometric Algebra for Physicists (2007)*; *Geometric Algebra for Computer Science (2007)*.

My *A Survey of Geometric Algebra and Geometric Calculus* provides an introduction for someone who already knows linear algebra. It contains a guide to further reading, online and off. It is available at this book's webpage.

Part I of this book is standard linear algebra. Part II introduces geometric algebra. Part III covers linear transformations and their geometric algebra extensions, called outermorphisms.

A majority of the topics in the traditional linear algebra course is treated. The major exception is algorithms. For example, the algorithm for inverting a matrix is not covered. The concept and applications of the inverse are important. They are used in many places in this book. But the algorithm to compute the inverse teaches little about the concept or its applications. Similar remarks apply to algorithms for row reduction, solving systems of linear equations, evaluating determinants, computing eigenvalues and eigenvectors, etc.

To me, the benefit/cost of including the algorithms is too low. I do not need them for the theoretical development. No one applies them by hand anymore – except for exercises in linear algebra textbooks! They take up a substantial fraction of the standard syllabus, time that can be better spent on other topics – like geometric algebra. Why teach them in an elementary linear algebra course?

Some exercises and problems in the text require the use of the free multiplatform Python module $\mathcal{GA}$lgebra. It is based on the Python *symbolic* computer algebra library SymPy (*Sym*bolic *Py*thon). The file $\mathcal{GA}$lgebraPrimer.pdf at the book's web site describes the installation and use of the module.

The book covers matrix arithmetic, the application of matrices to systems of linear equations, the matrix representation of linear transformations, the matrix version of the singular value decomposition, and several matrix applications. However, matrices play a smaller role than in most linear algebra texts. A major reason is that matrices are used in the omitted algorithms. Also, geometric algebra often replaces matrices with better alternatives. For example, the geometric algebra definition of a determinant is intuitive and simple and does not involve matrices. And geometric algebra provides better representations than matrices for important classes of linear transformations, as shown in the text for projections, rotations, reflections, and orthogonal and skew transformations.

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that students attempt them while reading the text. Then they immediately confront the concept and get feedback on their understanding. There are over 300 more challenging problems at the end of most sections.

The exercises replace the "worked examples" common in most mathematical texts, which serve as "templates" for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file "LAGA Instructor.pdf" at the book's web site. Take it for what it is worth.

There is plenty of material here for a one semester course. The actual text is only about 190 pages, rather short for a linear algebra text, much less for one incorporating geometric algebra. One reason is that I have tried to avoid the "bloated textbook syndrome". Another is that the exercises mean that a reader will spend more time per page than is usual in an elementary mathematics text.

An instructor should be wary of adopting a nonstandard text such as this for a course as fundamental as linear algebra. It might allay worries about this to know that this book can be used as a linear algebra text, without geometric algebra. Chapters 1-4 and Sections 8.1, 8.2, 9.1-9.4, and 9.7 use no geometric algebra.[2] They cover the majority of topics in the traditional linear algebra course, with the exception of the aforementioned algorithms and determinants. Thus an instructor can include geometric algebra as time permits, or teach a two track course, with some students studying geometric algebra and some not.

The first part of the index is a *symbol* index.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book's web site as appropriate.

Geometric calculus is a powerful extension of vector calculus, just as geometric algebra is a powerful extension of vector algebra. The divergence and Stokes' theorems are special cases of a very general theorem relating derivatives to integrals. Also, complex variable theory extends to arbitrary (even and odd) dimensions. I have published a sequel to this book, *Vector and Geometric Calculus*. That book's website is `http://faculty.luther.edu/~macdonal/vagc/`.

Acknowledgements. I thank Ian-Charles Coleman, Gabriel Demuth, Peeter Joot, Gez Keenan, Adem Semiz, Dr. Vijay Sonnad, Quirino M. Sugon Jr., and Ginanjar Utama for helpful comments. I thank Martin Barrett, Professor Philip Kuntz, James Murphy, Robert Rowley, and Professor John Synowiec for reading all/most of the text and providing extensive and helpful comments and advice.

I am especially grateful to Professor Leo Dorst for providing helpful expert commentary and to Allan Cortzen, who has improved this book in many ways, including providing better proofs of several theorems.

I also thank Alan Bromborsky, author of $\mathcal{GA}$lgebra, for making changes which make it more useful to readers of this book.

Finally, I thank Professor Kate Martinson for help with the cover design.

Printings

From time to time I issue new printings of this book, with corrections and improvements. The printing version is shown on the title page. They are listed below.

[2]Except for the definition of determinants (p. 157). Note however that determinants are not a prerequisite for anything important in the linear algebra part of the book. In particular, they are not used in the discussion of eigenvalues.

Second Printing. I thank George Craig, who asked several penetrating questions which have improved the exposition.

Fourth Printing. I have removed the old Appendix B, *Software*, which documented Alan Bromborsky's Python module $\mathcal{GA}$lgebra. The documentation there became out of date as the module improved.

A new Chapter 10 is devoted to the increasingly important *conformal model* of geometric algebra. It can be skipped to after reading Part II, *Geometric Algebra*, as the chapter does not reference Part III, *Linear Transformations*. I have written a Jupyter notebook cm3.ipynb based on $\mathcal{GA}$lgebra for making calculations in the 3D conformal model.

The current versions of GAlgebraPrimer.pdf and cm3.ipynb are available at the book's website and are bundled with the $\mathcal{GA}$lgebra distribution.

I give a special thanks to Gregory Grunberg, who has done much to improve this book, especially the new Chapter 10. The book is better for his efforts.

November 2019 Printing. Theorem 6.1-G6, G7 of the previous printing of have been replaced with a new G6, which better captures the idea of a k-vector. The definition of a blade has been changed to an equivalent one and moved to Section 6.1. As a result, Sections 6.1-6.5 have been rearranged.

April 2020 Printing. There are several improvements in the presentation.

October 2020 Printing. There are several improvements in the presentation.

June 2021 Printing. I thank J. M. Caillol, Pr, for helping me to get G6 of Theorem 6.1 right, and for suggesting Problem 6.2.9. The extended fundamental identity (Theorem 6.28) has been given the prominence it deserves. There are several improvements in the presentation. All errors known to me have been corrected.

Jan 2022 Printing. There are several improvements in the presentation. All errors known to me have been corrected.

August 2022 Printing. All errors known to me have been corrected.

> In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, ... Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius. – C. F. Gauss

To the Student

Linear algebra is indispensable in many disciplines, including mathematics, statistics, physics, computer science, chemistry, biology, engineering, and economics. Linear algebra is more widely used than any other college level mathematics, with the possible exception of calculus. You can see yourself that it is widely used: whenever a new concept is introduced in the text, Google it. You will find many links.

Most of the mathematics taught in single variable calculus courses has been known for 250 years. But mathematics is not a fixed body of knowledge, unchanged for hundreds of years. You are used to the fact that technology advances year by year. Mathematics also advances, though not as rapidly.

Linear algebra as we know it today is the result of a vast undertaking of abstraction, over centuries, unifying common aspects of many problems in many areas of mathematics and its applications. Do not translate "abstract" as "of no practical value": *abstraction gives linear algebra much of its practical power.* I hope that you will appreciate this by the time you finish the book.

The central theoretical importance of linear algebra started to be recognized early in the twentieth century. A sophomore linear algebra course has been part of the standard mathematics curriculum only since the early 1970's. The recent availability of cheap powerful computers has made it possible to solve more practical applications of linear algebra, causing an explosion of its use.

Geometric algebra is an extension of linear algebra. It originated in the 1960's and is under vigorous development today. It has found important applications in computer science, engineering, and physics. It is available to game developers for the Xbox and PlayStation video game consoles. This text is an attempt to keep up with these modern developments.

Most students find linear algebra hard, even many who have done well in previous mathematics courses. There are several reasons for this:

- Linear algebra has little connection to earlier courses. For example, this text makes only occasional, nonessential, reference to calculus.

- The large number of definitions and theorems can be overwhelming.

- Reasoning dominates calculation in linear algebra. The reasoning requires what has been called "a mathematical frame of mind". This is a new way of thinking, difficult to describe to those who have not acquired it.

How should you cope with these difficulties? Research clearly shows that *actively* engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

- Don't just read the text, *study* the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you understand.
- Instructors in your previous mathematics courses have probably urged you to try to *understand*, rather than simply memorize. That advice is especially appropriate for this text.
- Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand.
- Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.
- Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.
- Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Some are trivial, there just to make sure that you are paying attention. Do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!

  The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that "mathematical frame of mind".

  Write your solutions neatly in clear correct English.
- Proofs are important, but perhaps less so than the above. On a first reading, don't get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.
- Take the above points seriously!

Appendix A, *Prerequisites*, describes the mathematical background necessary to read this text. You might want to look it over now, to make sure that you are ready.

Some exercises and problems in the text require calculations unfeasible to perform by hand. GAlgebraPrimer.pdf, available at the book's webpage, describes how to install and use the computer algebra system $\mathcal{GA}$lgebra for this. It is written in Python, a free multiplatform language.

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