5.2 Oriented Solids

Definition 5.6 (Oriented solid)**.** An *oriented solid* $\mathbf{T}$ is an oriented segment of (i.e., solid in) three dimensional space. See Figure 5.11.

The volume of $\mathbf{T}$ is called its *norm*. It is denoted $|\mathbf{T}|$.

A small oriented circle on $\mathbf{T}$'s surface designates its orientation.[a]

[a]The circle must be small: an oriented equator on a sphere looks clockwise from one pole and counterclockwise from the other.

Solids with the same norm and orientation are considered equal. In other words, we ignore their shape and position.

As with areas, orientations of solids can only be compared: two solids have either the same or opposite orientations. There is no independent way to specify the orientation of one solid. For example, to specify the orientation of $\mathbf{T}$ in the figure as clockwise, one must compare to a clock.

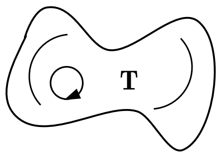

Fig. 5.11: An oriented solid.

Exercise 5.5. a. Define scalar multiplication $a\mathbf{T}$ of oriented solids.

b. Let $\mathbf{T}_0$ be a nonzero oriented solid. Show that every oriented solid $\mathbf{T}$ is a scalar multiple of $\mathbf{T}_0$: $\mathbf{T} = a\mathbf{T}_0$.

Two solids $a\mathbf{T}_0$ and $b\mathbf{T}_0$ have the same orientation if a and b have the same sign. Otherwise they have the opposite orientation.

c. Define addition $\mathbf{T}_1 + \mathbf{T}_2$ of oriented solids.

d. Define the zero oriented solid.

The exercise defines scalar multiplication, vector addition, and a zero for oriented solids. With these, oriented solids form a vector space. The next exercise asks you to verify the vector space Axiom V6 for oriented solids. We will not worry about the other axioms.

Exercise 5.6. Let $\mathbf{T}$ be an oriented solid. Show that $a(b\mathbf{T}) = (ab)\mathbf{T}$.

Exercise 5.5b shows that any nonzero oriented solid is a basis for the vector space of oriented solids. Therefore the vector space is one dimensional. We will see that this is an important fact. See Section 6.5.

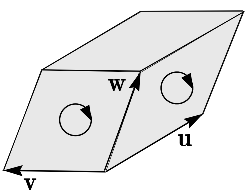

Fig. 5.12: The oriented solid $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$.

Oriented lengths $\mathbf{u}, \mathbf{v}, \mathbf{w}$, placed tail-to-tail, determine an oriented solid $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$, their outer product. See Figure 5.12. Rotate $\mathbf{u}, \mathbf{v}, \mathbf{w}$ cyclicly: $\mathbf{u}$ to $\mathbf{v}$, $\mathbf{v}$ to $\mathbf{w}$, and $\mathbf{w}$ to $\mathbf{u}$. All rotations are clockwise, as seen in the figure. Assign this orientation to the trivector, using Definition 5.6.

Exercise 5.7. Show that $\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}$ has the opposite orientation of $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$. Thus $\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = -(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})$.

In general, swapping *adjacent* vectors in an outer product negates the product.

Exercise 5.8. $\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u} = \pm(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})$. Which sign is correct? Why?

We have defined $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ but not $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ or $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$. Define them both equal to $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$. This makes the outer product associative: $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$.

The outer product in $\mathbb{G}^n$ is associative (Theorem 6.15c). Remember this important property of the outer product.

Perhaps you noticed that there are two ways to orient a segment of a line to form an oriented length, two ways to orient a segment of a plane to form an oriented area, and two ways to orient a segment of three dimensional space to form an oriented solid. This dichotomy extends to all dimensions.

Problems 5.2

5.2.1. Show geometrically that vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ are linearly independent if and only if $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \neq \mathbf{0}$. *Hint*: Recall Exercise 2.25.

5.2.2. Let

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \ \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3, \ \mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3.$$

Show that $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ is equal to

$$(u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3).$$

(Compare with Eq. (5.2).) *Hint*: Use Eq. (5.3) and the associativity of the outer product.

Computing $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$ gives the same result, consistent with associativity.

The problem shows that the outer product of three vectors is always a scalar multiple of $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. We already knew this from Exercise 5.5b. The scalar multiple is the oriented volume of the parallelepiped with sides $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Theorem 9.9 extends all this to $\mathbb{G}^n$.