Geometry without algebra is dumb! - Algebra without geometry is blind!
   - David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.
   - Allan McRobie and Joan Lasenby
To Ellen
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Preface

This text, *Vector and Geometric Calculus*, is intended for the second year vector calculus course. It is a sequel to my text *Linear and Geometric Algebra*. That text is a prerequisite for this one.

Linear algebra and vector calculus have provided the basic vocabulary of mathematics in dimensions greater than one for the past one hundred years. Geometric algebra generalizes linear algebra in powerful ways. Similarly, geometric calculus generalizes vector calculus in powerful ways.

Traditional vector calculus topics are covered here, as they must be, since readers will encounter them in other texts and out in the world.

The final chapter is a brief introduction to differential geometry, used today in many disciplines, including architecture, computer graphics, computer vision, econometrics, engineering, geology, image processing, and physics.

Vector algebra represents a plane in $\mathbb{R}^3$ with a vector orthogonal to the plane (a trick from the point of view of geometric algebra). But it cannot represent planes or higher dimensional geometric objects in $\mathbb{R}^n$, $n > 3$, a serious limitation. Geometric algebra represents the objects with multivectors.

Tensors and differential forms are two formalisms used to extend vector calculus to higher dimensions. Geometric calculus provides an at once simpler, more general, more powerful, and easier to grasp way to break loose from $\mathbb{R}^3$. Section 5.4, *Exact Fields*, translates elementary differential forms definitions, theorems, and examples to geometric calculus.

Linear algebra is the natural mathematical background for vector calculus. Yet even today it is unusual for a vector calculus text to have a linear algebra prerequisite. This has to do, I suppose, with publishers insisting that authors write to the largest possible audience. I cite my text *Linear and Geometric Algebra* freely and pervasively to advantage.

---

1D. Hestenes and G. Sobczyk have argued in detail the superiority of geometric calculus over differential forms (*Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht Holland 1984, Section 6.4, especially at the end.)
Vector and geometric algebra and also differential vector and geometric calculus (Part II of this book) are excellent places to help students better understand and appreciate rigor. But for integral calculus (Part III) rigorous proofs at the level of this book are mostly impossible. So I do not try.

Instead, I use the language of infinitesimals, while making it clear that they do not exist within the real number system. I believe that the first and most important way to understand integrals is intuitively: they “add infinitely many infinitesimal parts to give a whole”.

Others endorse this approach: “An approach based on [infinitesimals] closely reflects the way most scientists and engineers successfully use calculus.”\(^2\) From Lagrange: “When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results, ... we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.”\(^3\) And even Cauchy: “My main aim has been to reconcile the rigor which I have made a law in my Cours d’Analyse, with the simplicity that comes from the direct consideration of infinitely small quantities”\(^4\) (Emphasis added.)

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that students attempt them while reading the text. That way they immediately confront the concept and get feedback on their understanding. There are also more challenging problems at the end of most sections – almost 200 in all.

The exercises replace the “worked examples” common in most mathematical texts, which serve as “templates” for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Some exercises and problems require the use of the free multiplatform Python module G\(\text{A}\)lgebra. It is based on the Python symbolic computer algebra library SymPy (Symbolic Python). The file G\(\text{A}\)lgebraPrimer.pdf describes the installation and use of the module. It is available at the book’s web site.

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file “VAGC Instructor.pdf” at the book’s web site. Take it for what it is worth.

The first part of the index is a symbol index.

Some material which is difficult or less important is printed in this smaller font.


\(^4\)Quoted in *Cauchy’s Continuum*, Karin Katz and Mikhail Katz, Perspectives on Science 19, 426-452. Also at arXiv:1108.4201v2.
There are several appendices. Appendix A reviews some parts of Linear and Geometric Algebra used in this book. Appendix B provides a list of some geometric calculus formulas from this book. Appendix C provides a short comparison of differential forms and geometric calculus. Appendix D provides some technical results.

Numbered references to theorems, figures, etc. preceded by “LAGA” are to Linear and Geometric Algebra.

There are several URL's in the text. To save you typing, I have put them in a file “URLs.txt” at the book’s web site.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book’s web site as appropriate.

Acknowledgements. I thank Dr. Eric Chisolm, Greg Grunberg, Professor Philip Kuntz, James Murphy, and Professor John Synowiec for reading all/most of the text and providing and helpful comments and advice. Professor Mike Taylor answered several questions. I give special thanks to Greg Grunberg and James Murphy. Grunberg spotted many errors, made many valuable suggestions and is an eagle eyed proofreader. Murphy suggested major revisions in the ordering of my chapters.

I also thank the ever cooperative Alan Bromborsky for extending GA\textcopyright{}gebra to make it more useful to the readers of this book.

Thanks again to Professor Kate Martinson for help with the cover design.

In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, ... . Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius.

- C. F. Gauss
Printings

From time to time I issue new printings of this book, with corrections and minor improvements. The printing version is shown on the title page. I list below those with significant changes or with new helpful readers.

**Second Printing.** I thank again Gregory Grunberg for many suggestions and expert proofreading. Christoph Bader and Dr. Gavin Polhemous pointed out shortcomings in the notation of Section 5.5. And I thank Dr. Manuel Reenders, a recent arrival, for many suggestions and corrections, especially with regard to the exercises and problems.

**Third Printing.** I thank a new eagle eyed reader, Nicholas H. Okamoto, for sending me errata.

**Fifth Printing.** I thank the very careful new reader Professor Mark R. Treuden for helpful comments and corrections.

**August 2019 Printing.** A new Section 5.4, *Exact Fields*, translates differential forms language to geometric calculus language: closed fields, exact fields potentials, etc.. It was gathered and improved from existing sections.

**May 2020 Printing.** The notion of a tangent map has been moved to a more appropriate place. There are a few new exercises/problems. All errors known to me have been corrected. All were minor.
To the Student

Appendix A is a review of some items from Linear and Geometric Algebra (LAGA) used in this book. A quick read through it might be helpful before starting this book.

I repeat here my advice from Linear and Geometric Algebra.

Research clearly shows that actively engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

• **Read** Study the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you will understand.

• Instructors in your previous mathematics courses have probably urged you to try to understand, rather than simply memorize. That advice is especially appropriate for this text.

• Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand while you read.

• Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.

• Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.

• Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!
The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that “mathematical frame of mind”.

Write your solutions neatly in clear correct English.

- Proofs are important, but perhaps less so than the above. On a first reading, don’t get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.

- Take the above points seriously!

The World Wide Web makes it possible for me to leave out material that I would otherwise have to include. For example, the book refers to the *Coulomb force* without defining it. Perhaps you already know what it is. If not, and you want to know, actively engage the course material: Google it.
Part VI

Appendices
This appendix reviews some results from *Linear and Geometric Algebra* used in this book. The headings (in bold) are in the index.

**Geometric product.** (LAGA Theorem 6.1) The geometric product of multivectors $A$ and $B$ is written $AB$. For all scalars $a$ and all multivectors $A, B, C$:

- $G0$. $AB \in \mathbb{G}^n$.
- $G1$. $A(B + C) = AB + AC$, $(B + C)A = BA + CA$.
- $G2$. $(aA)B = A(aB) = a(AB)$.
- $G3$. $(AB)C = A(BC)$.
- $G4$. $1A = A1 = A$.
- $G5$. The geometric product of $\mathbb{G}^n$ is linked to the inner product of $\mathbb{R}^n$:
  \[ uu = u \cdot u = |u|^2 \quad \text{for all } u \in \mathbb{R}^n. \]

**Inner product.** (LAGA Definition 6.12) The inner product of a $j$-vector $A$ and $k$-vector $B$ is

\[ A \cdot B = \langle AB \rangle_{k-j}. \]

**Outer product.** (LAGA Definition 6.13) The outer product of a $j$-vector $A$ and $k$-vector $B$ is

\[ A \wedge B = \langle AB \rangle_{j+k}. \]

**Fundamental identities.** (LAGA Definition 5.9) For all vectors $a$ and $b$,

\[ ab = a \cdot b + a \wedge b \quad \text{(scalar + bivector)}. \]

For all vectors $a$ and all multivectors $B$, (LAGA Theorem 6.20)

\[ aB = a \cdot B + a \wedge B. \]
Reverse. (LAGA Definition 6.7) Let \( A = a_1 \cdots a_k \) be a geometric product of orthogonal vectors. Then the reverse of \( A \) is \( A^\dagger = a_k \cdots a_1 \).

Norm. (LAGA Definition 6.9) Expand a multivector \( A \) with respect to a canonical basis: \( A = \sum_j a_je_j \). (The sum is over members of a canonical basis of \( \mathbb{G}^n \), \( e_j \), times their scalar coefficients, \( a_j \). The upper case \( "J" \) emphasizes that the sum is over a multivector basis.) Then the norm \( |A| \) of \( A \) is defined by

\[
|A|^2 = \sum_j |a_j|^2.
\]

Unit pseudoscalar. (LAGA Definition 6.21) If \( \{e_1 \ldots e_n\} \) is an orthonormal basis for \( \mathbb{R}^n \), then the \( n \)-vector (and blade) \( I = e_1 \cdots e_n \) is a unit pseudoscalar. It satisfies \( |I| = 1 \). Also \( I^{-1} = (-1)^{(n-1)/2} I \). (LAGA Eq. (6.14)).

If \( v \) is a vector, then \( vI = (-1)^{(n-1)/2} Iv \). More generally, if \( B \) is a \( k \)-vector, then \( BI = (-1)^{(k-1)(n-1)/2} IB \). (LAGA Exercise 6.19)

Duality. (LAGA Definition 6.22) The dual of a multivector \( A \) is \( A^* = A/I \equiv AI^{-1} \), where \( I \) is the unit pseudoscalar. The dual satisfies (LAGA Theorem 6.24):

a. \( (aA)^* = aA^* \).

b. \( (A + B)^* = A^* + B^* \).

c. \( A^{**} = (-1)^{(n-1)/2} A \).

d. If \( A \) is a \( j \)-blade then \( A^* \) is an \( (n-j) \)-blade.

e. If \( A \) represents a subspace \( U \), then \( A^* \) represents \( U^\perp \).

f. \( |A^*| = |A| \).

g. If \( A \) is a \( j \)-vector, then \( A^* \) is an \( (n-j) \)-vector.

These duality relations are used often in this book (LAGA Theorem 6.26):

\[
(A \cdot B)^* = A \wedge B^*, \quad (A \wedge B)^* = A \cdot B^*.
\]

The formulas are easy to memorize. It is probably worthwhile for you to do so. In \( \mathbb{R}^3 \), \( v^{**} = -v \), because \( I^2 = -1 \).

Cross product. (LAGA Eq. (6.18)) The cross product is

\[
u \times v = (u \wedge v)^*.
\]

It is orthogonal to the plane of \( u \wedge v \), i.e., orthogonal to \( u \) and \( v \), in the direction given by the right hand rule. Its norm is \( |u \times v| = |u||v| \sin \theta \).

Linear transformation. (LAGA Definition 8.1) A function \( f: \mathbb{R}^m \to \mathbb{R}^n \) is a linear transformation if \( f(au) = af(u) \) and \( f(u + v) = f(u) + f(v) \).

Adjoint. (LAGA Theorem 8.2) The adjoint of a linear transformation \( f: U \to V \) is the unique linear transformation \( f^*: V \to U \) satisfying

\[
f(u) \cdot v = u \cdot f^*(v).
\]

Outermorphism. (LAGA Definition 8.18) Every linear transformation \( f: \mathbb{R}^n \to \mathbb{R}^m \) extends to an outermorphism \( f: \mathbb{G}^n \to \mathbb{G}^m \) satisfying \( f(A \wedge B) = f(A) \wedge f(B) \) for all multivectors \( A \) and \( B \).
Operator norm. (LAGA Problem 8.1.14) Let \( U \) and \( V \) be vector spaces. The set of all linear transformations \( f: U \to V \) is a vector space. Denote it \( \mathcal{L}(U, V) \). If \( U \) and \( V \) are inner product spaces, define the operator norm on \( \mathcal{L}(U, V) \):

\[
|f|_0 = \max_{|u| \leq 1} |f(u)|.
\]

The operator norm satisfies \(|f(u)| \leq |f|_0 |u|\) for all \( u \in U \).

Determinants. (LAGA Definition 8.4) If \( f \) is a linear transformation on \( \mathbb{R}^n \), then its determinant is defined by \( f(I) = \det(f)I \). It satisfies \( \det(g \circ f) = \det(g) \det(f) \). If \( f \) has an inverse, then \( \det(f^{-1}) = (\det(f))^{-1} \).

\[
\begin{align*}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= ad - bc. \\
\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} &= a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & c \\ g & h \end{bmatrix}.
\end{align*}
\]
Appendix B

Formulas

First order Taylor expansion, Eq. (3.20):
\[ f(x + h) = f(x) + \partial_i f(x)h_i + \frac{1}{2} \partial_{ij} f(x + t^* h)h_i h_j. \]

Differential, Eq. (3.5):
\[ f'(x)(h) = \partial_i f(x)h_i. \]

Jacobian matrix, Eq. (3.7):
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m}
\end{bmatrix} [h].
\]

Chain rule, Eq. (3.11): \((g \circ f)' = g' \circ f\). In detail: \((g \circ f)'_x = g'_f f'_x\) (Theorem 3.14).

Directional derivative, Eq. (3.9):
\[ \partial_h f(x) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}. \]

Differential \(\equiv\) Directional derivative, Eq. (3.10):
\[ \partial_h f(x) = f'_x(h). \]

Gradient, Eq. (5.1):
\[ \nabla F(x) = e_i \partial_i F(x). \]

Eq. (5.2): \(\partial_h f(x) = (h \cdot \nabla) f(x)\)

Gradient = Divergence + Curl, Eqs. (5.3):
\[ \nabla F = \nabla \cdot F + \nabla \wedge F. \]

Theorem 5.9:
  a. \(\nabla \wedge (\nabla \wedge F) = 0\). (The curl of a curl is zero.)
  b. \(\nabla \wedge (\nabla f) = 0\). (The curl of the gradient of a scalar field is zero.)
  c. \(\nabla \cdot (\nabla \cdot F) = 0\). (The divergence of a divergence is zero.)

Equations (5.7), (5.12), and (5.14):
\[
\begin{align*}
\nabla f &= \partial_1 f e_1 + \partial_2 f e_2 + \partial_3 f e_3 \\
\nabla \cdot f &= \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3 \\
\nabla \wedge f &= (\partial_1 f_2 - \partial_2 f_1) e_1 \wedge e_2 + (\partial_2 f_3 - \partial_3 f_2) e_2 \wedge e_3 + (\partial_3 f_1 - \partial_1 f_3) e_3 \wedge e_1
\end{align*}
\]
Curvilinear coordinate bases, Eq. (5.16):
\[ w_j = \frac{\partial x}{\partial w_j} = \frac{\partial x_i}{\partial w_j} e_i \quad \text{and} \quad w^k = \nabla w_k = \frac{\partial w_k}{\partial x_i} e_i. \]

Vector derivative operator, Eq. (5.23): \( \partial = x^u \partial_u + x^v \partial_v. \)

Path integral, Eq. (7.3): \( \int_C F \, ds = \int_{[a,b]} F(x(t)) |x'(t)| \, dt. \)

Line integral, Eq. (7.9): \( \int_C F \, ds = \int_{[a,b]} F(x(t)) \, x'(t) \, dt. \)

Change of variables, Eq. (8.5):
\[ \iint_A F(x,y) \, dA = \iint_{A^*} F(g(u,v)) \det(g'_{u,v}) \, dA^*. \]

Surface integral, Eq. (9.1): \( \iint_S F \, dS = \iint_A F(x(u,v)) |x_u(u,v) \wedge x_v(u,v) | \, dA. \)

Flux integral, Eq. (9.5): \( \iint_S F \, dS = \iint_A F(x(u,v)) (x_u(u,v) \wedge x_v(u,v) ) \, dA. \)

Corollary, Eq. (9.9): \( \iint_S f \cdot d\sigma = \iint_A f(x) \cdot (x_u \times x_v) \, dA. \)

Fundamental theorem, Eq. (10.1): \( \int_M d^m x \, \partial F = \oint_{\partial M} d^{m-1} x \, F. \)

Directed integral, Eq. (10.2): \( \int_M d^m x \, F = \int_A (x_{u_1} \cdots \wedge \cdots x_{u_m}) \, dA \, F. \)

Divergence (Gauss') theorem, Eq. (10.8): \( \iint_V \nabla \cdot f \, dV = \iint_S f \cdot d\sigma. \)

Curl (Stokes') theorem, Eq. (10.12): \( \iint_S (\nabla \times f) \cdot d\sigma = \oint_{\partial S} f \cdot ds. \)

Green's theorem, Eq. (10.15): \( \iint_R (\partial_x Q - \partial_y P) \, dA = \oint_C (P \, dx + Q \, dy). \)

Fundamental theorem for curves, Eq. (10.17): \( \int_C ds \, \partial F = F(x_2) - F(x_1). \)

Frenet-Serret equations, Eq. (11.1): \( \dot{t} = \kappa n, \quad \dot{n} = \tau b - \kappa t, \quad \dot{b} = -\tau n. \)

Darboux bivector, Eq. (11.3): \( \Omega = \frac{1}{2} (t \wedge \dot{t} + n \wedge \dot{n} + b \wedge \dot{b}) = \kappa t n + \tau n b. \)

Metric \( G, \) Eq. (11.5), and its inverse \( G^{-1}, \) Eq. (11.8):
\[ G = [g_{ij}] = [x_i \cdot x_j] = \begin{bmatrix} x_1 \cdot x_1 & x_1 \cdot x_2 \\ x_2 \cdot x_1 & x_2 \cdot x_2 \end{bmatrix} \]
\[ G^{-1} = [g^{ij}] = [x^i \cdot x^j] = \begin{bmatrix} x^1 \cdot x^1 & x^1 \cdot x^2 \\ x^2 \cdot x^1 & x^2 \cdot x^2 \end{bmatrix}. \]

Shape operator, Eq. (11.10): \( S_p(h) = -\partial_h \hat{n}(p). \)

Darboux basis, Eq. (11.12): \( \kappa n = \kappa n \hat{n} + \kappa n \times t. \)
Appendix C

Differential Forms

The theory of differential forms is a popular extension of vector calculus. Geometric calculus encompasses and improves this theory. Despite this, the transition from vector calculus to geometric calculus is easier than that from vector calculus to differential forms. This appendix compares the two formalisms in $\mathbb{R}^n$ for those familiar with differential forms. The comparison on manifolds is more complicated. I do not give it here.\footnote{For this, see D. Hestenes and G. Sobczyk, (Clifford Algebra to Geometric Calculus, D. Reidel, Dordrecht Holland 1984, Section 6.4.}

The curl $\partial \wedge$ operator is central in Sections 5.4 and 10.3. In $\mathbb{R}^n$, this geometric calculus operator corresponds to the exterior derivative $d$ operator of differential forms. The geometric calculus identity $\nabla \wedge (\nabla \wedge F) = 0$ (Theorem 5.9a) corresponds to the differential forms identity $d^2 = 0$.

The curl operates on $k$-vectors to produce $k+1$-vectors. The exterior derivative operates on $k$-forms to produce $k+1$-forms.

Besides $\partial \wedge$, geometric calculus has $\partial \cdot$ and their sum $\partial$, for which differential forms has no counterpart.

The theory of differential forms integrates scalar valued functions on a manifold $M$: $\int_M f (dx_1 \wedge \cdots \wedge dx_m)$. The result is a scalar. Geometric calculus integrates multivector valued functions on $M$ with a directed integral: $\int_M d^m x F = \int_M (dx_1 \wedge \cdots \wedge dx_m) F$. The result is a multivector. The infinitesimal $d^m x$ has a geometric interpretation as an infinitesimal pseudoscalar tangent to $M$. It can be manipulated with the operations of geometric algebra.

Section 10.5 indicated how geometric calculus extends complex variable theory naturally to all dimensions, even to manifolds. Differential forms provide no such extension. In particular, differential forms cannot express Cauchy’s integral theorem from standard complex variable theory as a single formula.
<table>
<thead>
<tr>
<th>Geometric calculus: $\partial \wedge$</th>
<th>Differential forms: $d$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>0-vector (scalar) field</strong></td>
<td><strong>0-form</strong></td>
</tr>
<tr>
<td>$\partial \wedge f = \partial_1 f e_1 + \partial_2 f e_2 + \partial_3 f e_3$</td>
<td>$df = \partial_1 f , dx_1 + \partial_2 f , dx_2 + \partial_3 f , dx_3$</td>
</tr>
<tr>
<td><strong>1-vector (vector) field</strong></td>
<td><strong>1-form</strong></td>
</tr>
<tr>
<td>$\partial \wedge (v_1 e_1 + v_2 e_2 + v_3 e_3)$</td>
<td>$d(v_1 , dx_1 + v_2 , dx_2 + v_3 , dx_3)$</td>
</tr>
<tr>
<td>$= (\partial_1 v_2 - \partial_2 v_1) e_1 \wedge e_2$</td>
<td>$= (\partial_1 v_2 - \partial_2 v_1) , dx_1 \wedge dx_2$</td>
</tr>
<tr>
<td>$+ (\partial_2 v_3 - \partial_3 v_2) e_2 \wedge e_3$</td>
<td>$+ (\partial_2 v_3 - \partial_3 v_2) , dx_2 \wedge dx_3$</td>
</tr>
<tr>
<td>$+ (\partial_3 v_1 - \partial_1 v_3) e_3 \wedge e_1$</td>
<td>$+ (\partial_3 v_1 - \partial_1 v_3) , dx_3 \wedge dx_1$</td>
</tr>
<tr>
<td><strong>2-vector (bivector) field</strong></td>
<td><strong>2-form</strong></td>
</tr>
<tr>
<td>$\partial \wedge (B_{12} e_1 \wedge e_2 + B_{23} e_2 \wedge e_3$</td>
<td>$d(B_{12} , dx_1 \wedge dx_2 + B_{23} , dx_2 \wedge dx_3$</td>
</tr>
<tr>
<td>$\quad + B_{31} e_3 \wedge e_1)$</td>
<td>$\quad + B_{31} , dx_3 \wedge dx_1)$</td>
</tr>
<tr>
<td>$= (\partial_1 B_{23} + \partial_2 B_{31} + \partial_3 B_{12})$</td>
<td>$= (\partial_1 B_{23} + \partial_2 B_{31} + \partial_3 B_{12})$</td>
</tr>
<tr>
<td>$\quad e_1 \wedge e_2 \wedge e_3$</td>
<td>$\quad dx_1 \wedge dx_2 \wedge dx_3$</td>
</tr>
<tr>
<td><strong>3-vector (trivector) field</strong></td>
<td><strong>3-form</strong></td>
</tr>
<tr>
<td>$\partial \wedge (T_{123} e_1 \wedge e_2 \wedge e_3) = 0$</td>
<td>$d(T_{123} , dx_1 \wedge dx_2 \wedge dx_3) = 0$</td>
</tr>
</tbody>
</table>

Correspondence between $\partial \wedge$ and $d$ in 3D cartesian coordinates.
Appendix D

Extend Fields on Manifolds

The technical results in this appendix are due to Professor Michael D. Taylor (private communication). They are cited a few times in the text. I cannot find them in the literature, so they are given here, for completeness. Most readers will not want to work through the proofs, simply accepting the results.

**Lemma** (Extend parameter functions.) Let \( x: A \subseteq \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3 \) parameterize a surface. Fix \((u_0, v_0) \in A\) and set \( p_0 = x(u_0, v_0) \). Embed \((u,v) \in \mathbb{R}^2\) in \( \mathbb{R}^3\): \((u,v,0)\). Then there is an open set \( U \) in \( \mathbb{R}^3\) containing \((u_0, v_0, 0)\), an open set \( V \subset \mathbb{R}^3\) containing \( p_0 \), and a continuously differentiable invertible function \( X: U \rightarrow V \) with \( X(u,v,0) = x(u,v) \). Furthermore, \( X^{-1} \) is continuously differentiable.

**Proof.** Extend the tangent plane basis \( \{ \partial_u x(u_0, v_0), \partial_v x(u_0, v_0) \} \) to an \( \mathbb{R}^3\) basis \( \{ \partial_u x(u_0, v_0), \partial_v x(u_0, v_0), e \} \). Define \( X(u,v,z) = x(u,v) + z e \). Then \( X(u,v,0) = x(u,v), \) as required. And \( X \) is, with \( x \), continuously differentiable:

\[
\partial_u X(u,v,z) = \partial_u x(u,v), \quad \partial_v X(u,v,z) = \partial_v x(u,v), \quad \partial_z X(u,v,z) = e.
\]

The three vectors form a basis at \((u_0,v_0,0)\), so \( X'(u_0,v_0,0) \) is invertible.

Now apply the Inverse Function Theorem 3.19. \( \Box \)

**Theorem** (Extend fields on manifolds). Let \( x(u,v) \) parameterize a surface \( S \subset \mathbb{R}^3\). Let \( F \) be a continuously differentiable multivector field on \( S \) (Definition 5.23). Then given \( p_0 \in S \), there is an open set \( V \) in \( \mathbb{R}^3\) containing \( p_0 \), and a continuously differentiable multivector field \( \bar{F} \) on \( V \) with \( \bar{F} = F \) on \( V \cap S \).

**Proof.** In the setup of the lemma define \( \pi(u,v,z) = (u,v) \). Then define \( \bar{F} = F \circ x \circ \pi \circ X^{-1} \). Since \( F \circ x, \pi, \) and \( X^{-1} \) are all continuously differentiable, so is \( \bar{F} \), by the chain rule. Let \( p \in V \cap S \) and \( x(u,v) = p \). Then

\[
\bar{F}(p) = (F \circ x \circ \pi \circ X^{-1})(p) = (F \circ x \circ \pi)(u,v,0) = (F \circ x)(u,v) = F(p). \quad \Box
\]

The lemma and theorem easily generalize to higher dimensions.

Appendix E

Errata
May 24, 2020

Errata for *Vector and Geometric Calculus*  
Printings 1-5

p. 81. Replace the multivector \( G \) with the vector \( g \) and the multivector \( \bar{G} \) with the vector \( \bar{g} \) everywhere. And in the statement of Theorem 5.27, \( \partial \cdot h \rightarrow h \cdot \partial \).

p. 155, Exercise 10.17. Replace with

Let \( F(x, y) = u(x, y) + v(x, y)i \), where \( u \) and \( v \) are scalar valued and \( i \) is the unit pseudoscalar of the \( xy \)-plane. Suppose that \( F \) is analytic. Prove the Cauchy-Riemann equations: \( u_x - v_y = 0 \) and \( v_x + u_y = 0 \).

Errata for *Vector and Geometric Calculus*  
Printings 1-4

Note: “p. m (n)” refers to page \( m \) of Printing 4 and page \( n \) of Printings 1-3.

p. 30, proof of Theorem 3.8. “the last sum in Eq (3.6) approaches 0.”  
→ “the right side of Step (3), divided by \( |h| \), approaches 0 with \( h \).”

p. 31 (29), just before Theorem 3.10. \( f'_x(h) = [f'_x][h] \rightarrow [f'_x(h)] = [f'_x][h] \).


p. 44, bottom. “for \( y(x) \) near \( a \)” → “for \( y \) in terms of \( x \) near \( a \).”

p. 56 (54), an omission, not an error. New second paragraph after Definition 4.5: “The tangent space is a vector space (LAGA Exercise 8.12).”

p. 58 (55), caption of Figure 5.14. “\( f' \) maps” → “\( f'_x \) maps”.

p. 62 (58), Definition 5.2. Add a footnote after the first line: “i.e., the scalar coefficients of \( F \) are differentiable.”

p. 64, Problem 5.2.10. Replace “\( f'_x(a) \)” with “\( f'_x(a) \)”.

p. 79 (73), Problem 5.5.1a. “Eq. (6.25)” → “Eq. (5.19)”.

p. 85, first line of the proof of Theorem 6.5, improved: “First, \( \partial_i f(x)h_i = \nabla f(x) \cdot h = 0 \).”

p. 86, Definition 6.3. In “a strict local minimum at \( x \) if \( f(x) \leq f(x + h) \)” change to “\( f(x) < f(x + h) \).”

p. 96, Exercise 7.1. The approximate value of the integral is 0.2570120954.

p. 99, Problem 7.1.1. At end: “Parts (c)-(d)”.

p. 105. Remove Problem 7.2.9. It is a duplicate of Problem 7.2.3.

p. 115, Problem 7.2.3. Integral should read \( \int_{[a, b]} \sqrt{r^2 + r^2} \, d\theta \).

p. 129, just before Figures 8.2-8.4. Strike “to a scalar”. 
p. 131, Problem 9.1.4. Displayed equation should read $2\pi \int_a^b \sqrt{1 + r'(z)^2} \, r(z) \, dz$.

p. 132. The formula for area in the middle of the page should be $\iint_A |x_u(u,v) \wedge x_v(u,v)| \, dA$.

p. 138, Problem 9.2.2. The answer is $-4\pi/3$.

p. 142, last paragraph, first sentence. “multiple integrals” → “directed integrals”.

p. 144, Exercise 10.2. Replace the hint with “Fact: The boundary of a boundary is empty: $\partial(\partial M) = \emptyset$.”

Corollary 10.3 (Generalized divergence theorem). Let $M$ be a bounded $m$-dimensional manifold in $\mathbb{R}^m$ and $\mathbf{n}$ be its outward normal. Let $F$ be a multivector field on $M$. Set $d\sigma = \mathbf{n} d^{m-1} x$. Then

$$\int_M \nabla \cdot F \, d^m x = \oint_{\partial M} d\sigma \cdot F.$$ 

Proof. Suppose first that $F$ is of a single grade, $g$. Use the extended fundamental identity (LAGA Theorem 6.28) to split the geometric product in both integrands of Eq. (10.5) into inner and outer products, of grades $g-1$ and $g+1$ respectively. Equating the inner product parts gives Eq. (10.7) for a single grade $F$.

To finish, apply this to each grade of a multigrade $F$ and add. □

p. 153, first displayed equation: $(\nabla \times f)(x^*) \cdot \mathbf{n} \rightarrow ((\nabla \times f) \cdot \mathbf{n})(x^*)$. And the left side of Eq. (10.14) should read: $((\nabla \times f) \cdot \mathbf{n})(x_0)$.

p. 166, Theorem 11.8. “Let $\mathbf{x}(u(t), v(t))$” → “Let $\mathbf{x}(u_1(t), u_2(t))$”.
Errata for *Vector and Geometric Calculus*
Printings 1-3

p. 30, Theorem 3.10, first line of proof:

\[(\partial_i f_1(x), \ldots, \partial_i f_n(x)) \rightarrow (\partial_i f_1(x), \ldots, \partial_i f_m(x)).\]

p. 40, just before the problems. "series of a vector valued function f centered" \(\rightarrow\) "series of an f centered".

p. 44, first line. "differentiable" \(\rightarrow\) "continuously differentiable"

p. 51, Theorem 4.3 should read:

Let \(x_1(t)\) and \(x_2(t)\) parameterize curves and suppose that \(x_1'(t)\) and \(x_2'(t)\) exist. Then \((x_1(t)x_2(t))'\) exists and

\[(x_1(t)x_2(t))' = x_1'(t)x_2(t) + x_1(t)x_2'(t).\]

Change Eq. (4.6) similarly. Remove the paragraph following the theorem.

p. 55, change bottom to

Let \(f\) be a 1-1 map between manifolds of equal dimension.

Then the tangent map \(f'_p\) maps \(T_p\) to \(T_{f(p)}\).

p. 58. Remove Problem 4.3.6 Part (b) and restate Part (a):

The definition of the determinant of a linear transformation on a vector space (LAGA Definition 8.21) does not apply to the linear transformation \(f'_p\) because it is between two vector spaces. Define \(\text{det}(f'_p)\). Hint: See LAGA Definition 8.21.

p. 64, Problem 5.2.11. \(\nabla \wedge e = -\partial_t B \rightarrow \nabla \wedge e = \partial_t B.\)

p. 76, Problem 11.2.7. New Part (b): Define a directional derivative for fields defined on a surface by \(\partial_h f(p) = (h \cdot \partial)f(p)\) (Definition 11.10). Compute \(\partial_t t\) on the equator. Ans. \(-\sin \theta i + \cos \theta j)/\rho.\)

Note that at the equator \(t\) is in the tangent plane but \(\partial_t t\) is not.

p. 82, Theorem 6.7 statement. \(g(x_0) = c \rightarrow g(x) = c.\)

p. 105, Problem 5.4.1. Field should be \(e^x(\sin(xy) + y \cos(xy))i + xe^x \cos(xy)j\)

p. 116, Exercise 8.8. The answer is \(\pi(e^4 - 1).\)

p. 132, Corollary 10.3. "Let \(f\) be a multivector field" \(\rightarrow\) "Let \(f\) be a vector field"

p. 134, Problem 10.2.6c. \(\oint_C e \cdot ds = \partial_t \int_S B \cdot dS \rightarrow \oint_C e \cdot ds = -\partial_t \int_S B \cdot dS\)

p. 186. Add "65" and "66" to gradient entry. Add "98" to curl entry. Add "124" to divergence entry.
Errata for *Vector and Geometric Calculus*
Printings 1-2

p. 14, line 8: Delete “in”.

p. 23, line 4: Change the period after “tangents” into a comma.

p. 28, Exercise 3.9: Change “Eq. (3.23)” to “Eq. (3.18)”.  

\[ f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \to f : \mathbb{R}^n \to \mathbb{R}^m. \]

p. 38, Problem 3.4.12c. Change to “The ideal gas law is \( p v = nRT \), where \( n, p, v, T \) are the number of moles, pressure . . . .”

p. 43, last two lines.  
\[ f(a, b) = 0 \to f(a, b) = (2, 2). \]

p. 49, Corollary 5.7.  
\[ f'(*) (b) \to f'_x(*) (b). \]

p. 55, Figure 5.14, caption. “onto \( S \)” \to “to \( S \)”

p. 59, line -5: Change “Eq. (5.2) and Eq. (3.6)” to “Eq. (5.2) and Eq. (3.23)”.  

p. 64, bottom. Problem 4.3.12 \( \to \) LAGA Problem 4.3.12

p. 69, Eq. (5.17).  
\[ \frac{\partial x_i}{\partial w_j} e_i \to \frac{\partial x_i}{\partial w_j} e_i. \]

p. 72, Exercise 5.36. Equations should read  
\[ \hat{\phi} = \cos \phi (\cos \theta i + \sin \theta j) - \sin \phi k, \quad \hat{\theta} = -\sin \theta i + \cos \theta j. \]

p. 75, footnote.  
\( \mathbb{R}^3 \to \mathbb{R}^n. \)

p. 76, Problem 5.6.1b, second printing only. “even though \( \hat{t} \) is” \to “even though \( \hat{t} \) is”.

p. 79, Theorem 6.5. A much better proof:  

**Proof.** Since \( \nabla f(x) = 0 \), \( \partial_i f(x) h_i = 0 \). And \( \partial_i j f(x) h_i h_j > 0 \) for \( h \neq 0 \), since \( H f(x) \) is positive definite. Then \( \partial_i j f(x + t^* h) h_i h_j > 0 \) for small \( t^* h \neq 0 \), since the partial derivatives are continuous at \( x \). The theorem now follows from Eq. 3.2. \( \square \)

p. 81, Problem 6.1.2c.  
\[ \lim_{(x,y) \to \infty} \to \lim_{(x,y) \to \infty} f(x, y). \]

p. 90, equation mid-page,  
\[ \int_{[a, b]} af dx = a \int_{[a, b]} f dx \to \int_{[a, b]} cf dx = c \int_{[a, b]} f dx. \]

p. 91, after the sentence beginning with Think of. “Divide \( C \) into infinitesimal parts. Multiply the value of \( F \) on each part by the infinitesimal length \( ds \) of the part. Add to form the integral.”

p. 96, note at the bottom of the page. “here here” \to “here”.

p. 98, Problem ??c. Misplaced “): “\( F(x(u,v)x_u(u,v)) \)” \to “\( F(x(u,v))x_u(u,v) \)”.

p. 100, Exercise ?? “Theorem 7.11” \to “Theorem 7.10”.

p. 100, proof of Theorem 7.13: ‘it independent’ \to “it is independent”.

p. 100, proof of Theorem 7.13: ‘it independent’ \to “it is independent”.
p. 101, Exercise 5.20b. “not conservative” → “not conservative in \( \mathbb{R}^2 - \{0\} \)

p. 102, following Definition 5.18. “All simple closed curves” → “All closed curves”.

p. 103, line -7. Switch “m” and “M”.

p. 103, line -5: Change “Eq. (7.10)’” to “(Eq. (7.14))’.

p. 107, line -3: Change “set open” to “open set”.

p. 112, Exercise 8.2. \( \int_{\nu=0}^{1} \rightarrow \int_{\nu=0}^{2} \)

p. 124, Problem 9.2.3: “scalar + trivector” → “vector + trivector”.

p. 131, Fig. 10.6: Arrows should be reversed, as the \( M_i \) are “oriented clockwise”.

p. 136, second line of the proof of Corollary 10.5: \((-1)^{2 \times 2} \rightarrow (-1)^{2 \times 1} \)

p. 142, below Corollary 10.10. \( f(x) = \int_{a}^{x} f'(t)dt + f(a) \)

p. 146, lines 12 and 17: Change “Theorem 4.3b” and “Theorem 4.3” (both) to “Eq. (4.6)”.

p. 148, Theorem 11.5, Proof. \( \ddot{x} \rightarrow \dot{x} \), twice

p. 151, line above Def 11.10: Change “Definition 5.23” to “Definition 5.15”.

p. 158, middle displayed line in the proof of Theorem 11.25. Drop the middle term.


p. 162, line 5. Delete “is”.
Errata for *Vector and Geometric Calculus*

Printing 1

Due to publisher error the shading in several of figures is washed out in some copies of the book. The correct shading is shown below. I think that if Figure 2.1 is OK in your book, then all figures are.

2.1

2.2

2.3

2.6

8.1

8.2

8.3

8.4

8.5

8.6

9.5

p. 4, Exercise 1.1. “$c_{ij} \rightarrow c_{ik}$”.

p. 6, Problem 1.1.1. “$x(t) \rightarrow x(\theta)$”.
p. 8, Exercise 1.11. Delete “We will do this often.” Add “This will allow us to specialize formulas for surfaces defined parametrically to surfaces defined by $z = f(x, y)$. Exercise 5.38 is an example.”

p. 11, Exercise 1.14. “$x \neq 0$” → “$x > 0$”.


\[ \phi = \arccos(z/r) \rightarrow \phi = \arctan(r/z). \]

p. 22, bottom. Remove “The definition shows that $\partial_i F$ has the same grades as $F$. Parts disappear if their partial derivative is zero.”

p. 23, Exercise 3.1. “RE” → “$R^m$ to $R^n$.”

p. 33, Problem 3.2.1. Append the sentence “Then for fixed $x$, the differential is the linear transformation $h \mapsto f'(x)h$.”

p. 33, Problem 3.2.3. $(\rho, \theta, \phi) \rightarrow (\rho, \phi, \theta)$

p. 34, first and second displayed equations should read

\[
x, h \in \mathbb{R}^n \Rightarrow (g \circ f)(x) \in \mathbb{R}^p \Rightarrow (g \circ f)'(h) \in \mathbb{R}^p,
\]

\[
x, h \in \mathbb{R}^n \Rightarrow f'_x(h) \in \mathbb{R}^m \Rightarrow (g_{f(x)} \circ f'_x)(h) \in \mathbb{R}^p.
\]

p. 34. Line should read $4 \ldots + [g'(R(h)) |h| + S(kh)|k_h|].$

p. 34. Replace the end of the page with the following:
The added phrase “divided by $|h|$” is the reason for the changes. To finish, we show that the term in brackets above, divided by $|h|$, approaches zero with $|h|$. First, using the continuity of $g'$ (Theorem 2.10),

\[
\lim_{h \to 0} g'(R(h)) = g'(\lim_{h \to 0} R(h)) = g'(0) = 0.
\]

Second, with $|f'|_O$ the operator norm of $f'$,

\[
|h_k| \leq |f'(h)| + |R(h)||h| \leq |f'|_O|h| + |R(h)||h|.
\]

Thus, since $(h \to 0) \Rightarrow (k_h \to 0) \Rightarrow (S(k_h) \to 0), \lim_{h \to 0} |S(k_h)||k_h|/|h| = 0.$

p. 37, statement of Theorem 3.16.

“Then the inverse function $(f'_x)^{-1}$” → “Then the inverse function $f^{-1}$."

p. 40, following Theorem 3.13: “In other words, $\partial_h f$ is linear in both $h$ and $f$.”

p. 40, Problem 3.3.5. The variable names I used lead to confusion. Change to $f(x, y) = (x \cos y, x \sin y)$. And add “(All coordinates are cartesian.)”

p. 40, Problem 3.3.3. “continuity of $f$ at $x$.” → “continuity of $f$ at $x$."

p. 41, Statement of Theorem 3.19.

“has a differentiable inverse” → “has a continuously differentiable inverse”."

p. 42, third line. Remove “there is a neighborhood of each $y_i$ in which”.

p. 45, Exercise 3.6.1b. “Determine $\partial \rho/\partial x$.”

p. 48, below Eq. (4.3). “of higher dimension” → “in higher dimensions”. 
pp. 49-55. Equation (4.4) in Section 4.1 established the notation of $m$-dimensional manifolds $M$ as subsets of $\mathbb{R}^n$. However, Sections 4.2 and 4.3, while mostly internally consistent, are inconsistent with this notation. The following changes remove the inconsistency:

- p. 49, 2nd paragraph: “a curve $C$ in $\mathbb{R}^n$.”
- p. 50, Theorem 4.2: $m$ to $n$.
- p. 51, Theorem 4.4: $\mathbb{R}^m$ to $\mathbb{R}^n$.
- p. 52, Theorem: $\mathbb{R}^m$ to $\mathbb{R}^n$, $\mathbb{R}^n$ to $\mathbb{R}^n$.
- p. 53, First sentence and left part of Figure 4.5: $\mathbb{R}^m$ to $\mathbb{R}^n$.
- p. 53, Definition 4.5: $\mathbb{R}^m$ to $\mathbb{R}^n$.
- p. 53, line following Eq. (4.11): $m$ to $n$.
- p. 55, Theorem: $\mathbb{R}^m$ to $\mathbb{R}^n$, $\mathbb{R}^n$ to $\mathbb{R}^n$.

p. 49, Paragraph 2. “Let $x(t)$ parameterize a curve $C$. Then $x$ has a nonzero differential (p. 49). By Problem 3.2.1 it is . . . ”

p. 50, line after Theorem 4.3. “Definition 4.5” $\rightarrow$ “Eq. (4.5)”

p. 52, Theorem. “$f'_p$ is one-to-one.” $\rightarrow$ “$f'_p$ restricted to $T_p$ is one-to-one.”

p. 52, Problem 4.2.3.
  a. Show that components of $\Omega$ must have grade 2 or 3.
  b. Show that components of $\Omega$ must have grade 0 or 2.

p. 53, Eq. (4.10). “$\lim_{h \to 0}$” $\rightarrow$ “$\lim_{h \to 0}$”

p. 53, sentence below Definition 4.5. “Recall that the differential $x'_q$ is one-to-one and maps linearly independent vectors ... ”

p. 55, Theorem. “$f'_p$ is one-to-one.” $\rightarrow$ “$f'_p$ restricted to $T_p$ is one-to-one.”

p. 56, Problem 4.3.1b. $x_u \wedge x_v \rightarrow x_\phi \wedge x_\theta$.

p. 57, Definition 5.1. “Let $M$ be a manifold in $\mathbb{R}^n$. A field on $M$ is a function defined on $M$ whose values are in $\mathbb{G}^n$."

p. 62, Problem 5.2.5. Change to $\nabla \cdot (xf(|x|)) = nf(|x|) + |x|f'(|x|)$.

p. 64, Problem 5.2.11b. Remove the word “both”.

p. 67, Exercise 5.14a. $\nabla \cdot f = \partial_1 f_1 + \partial_2 f_2$.

p. 69, Exercise 5.17b. Printing 1 only. Replace “The bases $\{w_r(r, \theta)\}$ and $\{w_\theta(r, \theta)\}$ are not in general orthogonal” with “In general, neither $\{w_r(r, \theta)\}$ nor $\{w_\theta(r, \theta)\}$ is an orthogonal basis.”

p. 70, Exercise 5.33. “Hint: For Part (a) use Exercises 5.30 and 5.31.”

p. 71, Figure 5.11. “$x(c_1, c_2, c_3)$” $\rightarrow$ “$x(c_1, c_2, u_3)$”

p. 71, Paragraph 4. In general: (i). Each basis vector $x^k$ is orthogonal to the surface formed by fixing the coordinate $u_k$. (ii). Each basis vector $x_j$ is tangent to the curve which is the intersection of the two surfaces formed by fixing in turn the coordinates other than $u_j$.

p. 72, formula for $\nabla f$ in cylindrical coordinates. “$r^{-1} \partial r f_\theta$” $\rightarrow$ “$r^{-1} \partial \theta f$”.

p. 74, Problem 5.5.2. Better: “Hint: If $B$ is a blade, then $B^{-1} = B/B^2$, where $B^2$ is a scalar.”
p. 74, Problem 5.4.5. Delete. Renumber Problems 5.4.6-5.4.8 to 5.4.5-5.4.7.

p. 75, line 3. "manifolds in $\mathbb{R}^m$" → "manifolds in $\mathbb{R}^n$".

p. 76, Problem 5.6.1. "to the unit sphere" → "in $\mathbb{R}^2$"
"$D = \Pi_T(\partial)$" → "$D F = \Pi_T(\partial F)$".

p. 81, Problem 6.1.4. \[ m \quad \rightarrow \quad \bar{m} \]

p. 82, Theorem 6.7, proof.
"Let $x(\xi)$ parameterize ... $x(t) = x(\xi(t))$ parameterize a curve " →
"Let $x(t)$ be a parameterized curve".

p. 83, Problem 6.2.3. "on the triangle" → "inside the triangle".

Ans. Max 20, Min 4.

p. 89, last line of the displayed equation in the proof of Theorem 7.3
$|P| \rightarrow 0 \Rightarrow |P| \rightarrow 0 \rightarrow$.

p. 95, Definition 7.8. "tangent line to $S$" → "tangent line to $C$".

p. 122, Corollary 9.4, statement. "$S \subset \mathbb{R}^n$" → "$S \subset \mathbb{R}^3$".

p. 130, Exercise 10.2.
"defined on the boundary" → "defined on $M$ and $\partial M$".

p. 133, toward bottom. "which is wanting in the definition given by Eq. (5.3)."

p. 134, Problem 10.2.4b. Change vector field $f$ to scalar field $f$. Drop Part c.

p. 136, Corollary 10.5, Proof. "Step (2) uses LAGA Theorem 6.30c and $dS^* = d\sigma$." → "Step (2) uses LAGA Theorem 6.30c."

p. 138, toward bottom.
"which is wanting in the definition given by Eq. (5.3)."

p. 141, top. "manifolds of arbitrary dimension." → "$\mathbb{R}^m$."
p. 148, Theorem 11.5, first sentence of statement. Change to "Let \( x(s) \) and \( \bar{x}(s) \), \( 0 \leq s \leq L \), parameterize curves \( C \) and \( \bar{C} \)."

p. 150, Theorem 11.8, proof. \( \ell(C) = \int_{[a,b]} |x'(u_1(t), u_2(t))| \, dt \)

\[ |x'(u_1, u_2)|^2 = x'(u_1, u_2) \cdot x'(u_1, u_2) = \]

p. 152, Theorem 5.25, proof. \( f'(h) \rightarrow f'_p(h) \).

p. 152, Exercise 11.24a. Show that the metric \( G(r, \theta) = [r \ 0 \ 0 \ 1] \).

p. 161, Figure 11.3. Remove the hat on \( p_1 \) and \( p_2 \).

p. 163, Standard Terminology. "The metric \( G \) (Eq. (11.5))" \( \rightarrow \) "The expression \( ds^2 = g_{ij} du_i du_j \) (Eq. (11.7))".

p. 172, Differentiation entry. "print diff(diff(x**2,x),y)" \( \rightarrow \) "print diff(diff(y*x**2,x),y)".

p. 172, Jacobian entry. Redo: Jacobian. Let \( X \) be an \( m \times 1 \) matrix of \( m \) variables. Let \( Y \) be an \( n \times 1 \) matrix of functions of the \( m \) variables. These define a function \( f: X \in \mathbb{R}^m \mapsto Y \in \mathbb{R}^n \). Then \( Y.jacobian(X) \) is the \( n \times m \) matrix of \( f'_x \), the differential of \( f \).

\[ r, \theta = symbols('r \ theta') \]
\[ X = Matrix([r, \theta]) \]
\[ Y = Matrix([r*cos(\theta), r*sin(\theta)]) \]
\[ print Y.jacobian(X) \] # Print 2 \times 2 Jacobian matrix.
\[ print Y.jacobian(X).det() \] # Print Jacobian determinant (only if \( m = n \)).

Sometimes you want to differentiate \( Y \) only with respect to some of the variables in \( X \), for example when applying Eq. (3.24). Then include only those variables in \( X \). For example, using \( X = Matrix([r]) \) in the example above produces the 2 \times 1 matrix \( \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \).

p. 172, Iterated Integrals entry.
"make.symbols('x y')" \( \rightarrow \) "x, y = symbols('x y')".

p. 174, Reciprocal Basis entry. Drop everything after the first sentence.

p. 175. Change to
Compute the vector derivative \( \partial f \), divergence \( \partial \cdot f \), curl \( \partial \wedge f \):
\[ M.grad * f, \ M.grad < f, \ M.grad \wedge f. \]
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