

Vector and Geometric Calculus

September 2023 printing

Alan Macdonald

Professor Emeritus of Mathematics
Luther College, Decorah, IA 52101 USA
macdonal@luther.edu
faculty.luther.edu/~macdonal



Geometry without algebra is dumb! - Algebra without geometry is blind!

- David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.

- Allan McRobie and Joan Lasenby

To Ellen

Copyright © 2012
Alan Macdonald

Contents

Contents	iii
Preface	vii
To the Student	xi
I Preliminaries	1
1 Curve and Surface Representations	3
1.1 Curve Representations	5
1.2 Surface Representations	7
1.3 Polar, Cylindrical, Spherical Coordinates	11
2 Limits and Continuity	13
2.1 Open and Closed Sets	13
2.2 Limits	15
2.3 Continuity	18
II Derivatives	21
3 The Differential	23
3.1 The Partial Derivative	23
3.2 The Differential	28
3.3 The Directional Derivative	33
3.4 The Chain Rule	35
3.5 Taylor's Formula	40
3.6 Inverse and Implicit Functions	42
4 Tangent Spaces	47
4.1 Manifolds	47
4.2 Tangent Spaces to Curves	50
4.3 Tangent Spaces to Surfaces	54

5	The Gradient	59
5.1	Fields	59
5.2	The Gradient	60
5.3	Scalar and Vector Fields	67
5.4	Exact Fields	72
5.5	Curvilinear Coordinates	80
5.6	The Vector Derivative	87
6	Extrema	93
6.1	Extrema	93
6.2	Constrained Extrema	98
III	Integrals	103
7	Integrals over Curves	105
7.1	The Scalar Integral	105
7.2	The Path Integral	110
7.3	The Line Integral	114
8	Multiple Integrals	121
8.1	Multiple Integrals	121
8.2	Change of Variables	127
9	Integrals over Surfaces	131
9.1	The Surface Integral	132
9.2	The Flux Integral	134
IV	The Fundamental Theorem of Calculus	139
10	The Fundamental Theorem of Calculus	141
10.1	The Fundamental Theorem of Calculus	141
10.2	The Divergence Theorem	147
10.3	The Curl Theorem	151
10.4	The Gradient Theorem	156
10.5	Analytic Functions	157
V	Differential Geometry	161
11	Differential Geometry in \mathbb{R}^3	163
11.1	Curves	163
11.2	Surfaces	168
11.3	Curves in Surfaces	177
11.4	Differential Geometry in \mathbb{R}^n	183

VI Appendices	185
A Geometric Algebra Review	187
B Formulas from this Book	190
C Differential Forms	192
D Extend Fields on Manifolds	194
Index	195

Preface

Vector and Geometric Calculus is intended for the second year vector calculus course. It is a sequel to my text *Linear and Geometric Algebra*. That text is a prerequisite for this one. Single variable calculus is also a prerequisite.

Linear algebra and vector calculus have provided the basic vocabulary of mathematics in dimensions greater than one for the past one hundred years. Geometric algebra generalizes linear algebra in powerful ways. Similarly, geometric calculus generalizes vector calculus in powerful ways.

Traditional vector calculus topics are covered here, as they must be, since readers will encounter them in other texts and out in the world.

The final chapter is a brief introduction to (mostly 3D) differential geometry, used today in many disciplines, including architecture, computer graphics, computer vision, econometrics, engineering, geology, image processing, and physics.

Tensor calculus and differential forms are two formalisms used to extend vector calculus beyond three dimensions. Geometric calculus provides an at once simpler, more general, more powerful, and easier to grasp way to break loose from \mathbb{R}^3 .¹ Section 5.4, *Exact Fields*, translates elementary differential forms definitions, theorems, and examples to geometric calculus.

Linear algebra is the natural mathematical background for vector calculus. Yet even today it is unusual for a vector calculus text to have a linear algebra prerequisite. This has to do, I suppose, with authors and publishers wanting to reach the largest possible audience. I cite my text *Linear and Geometric Algebra* freely and pervasively to advantage.

Vector and geometric algebra and differential vector and geometric calculus (Part II of this book) are excellent places to help students better understand and create proofs. But for integral calculus (Part III) rigorous proofs of fundamental theorems at the level of this book are mostly impossible. So I do not try.

Instead, I use the language of infinitesimals, while making it clear that they do not exist within the real number system. I believe that the first and most important way to understand integrals is intuitively: they “add infinitely many infinitesimal parts to give a whole”. Rigorous definitions should come later.

¹D. Hestenes and G. Sobczyk have argued in detail the superiority of geometric calculus over differential forms (*Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht Holland 1984, Section 6.4, especially at the end.

Others endorse this approach: “An approach based on [infinitesimals] closely reflects the way most scientists and engineers successfully use calculus. We continue to find it remarkable that the mainstream mathematics community insists on downplaying the use of infinitesimals, most especially when teaching calculus.”² “The fact is that in many situations ... the interpretation of the integral as a sum of infinitesimals is the clearest way to understand what is going on.”³ From Lagrange: “When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results, ... we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.”⁴ And even Cauchy: “My main aim has been to *reconcile* the rigor which I have made a law in my Cours d’Analyse, with the simplicity that comes from the direct consideration of infinitely small quantities”⁵ (My emphasis.)

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that readers attempt them while reading the text. That way they immediately confront the concept and get feedback on their understanding. There are also more challenging problems at the end of most sections – almost 200 in all.

The exercises replace the “worked examples” common in most mathematical texts, which serve as “templates” for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Some exercises and problems require the use of the free multiplatform Python module `GAlgebra`. It is based on the Python symbolic computer algebra library SymPy (Symbolic Python). `GAlgebraPrimer.pdf` describes the installation and use of the module. `GAlgebra` is available at the book’s web site.

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file “VAGC Instructor.pdf” at the book’s web site. Take it for what it is worth.

The first part of the index is a symbol index.

Some material which is difficult or less important is printed in this smaller font.

²Tevian Dray and Corinne Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, The College Mathematics Journal **34**, 283-290 (2003).

³Gerald Folland, *Advanced Calculus*, p. 157, Pearson (2001).

⁴*Mécanique Analytique*, Preface; Ouvres, t. 2 (Paris, 1988), p. 14.

⁵Quoted in *Cauchy’s Continuum*, Karin Katz and Mikhail Katz, *Perspectives on Science* **19**, 426-452. Also at arXiv:1108.4201v2.

There are several appendices. Appendix A reviews some parts of *Linear and Geometric Algebra* used in this book. Appendix B provides a list of some geometric calculus formulas from this book. Appendix C provides a short comparison of differential forms and geometric calculus. Appendix D proves a couple of technical results needed in the text.

Numbered references to theorems, figures, etc. preceded by “LAGA” are to *Linear and Geometric Algebra*.

There are several URL’s in the text. To save you typing, I have put them in a file “URLs.txt” at the book’s web site.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book’s web site as appropriate.

Acknowledgements. I thank Dr. Eric Chisolm, Greg Grunberg, Professor Philip Kuntz, James Murphy, and Professor John Synowiec for reading all/most of the text and providing helpful comments and advice. Professor Mike Taylor answered several questions. I give special thanks to Greg Grunberg and James Murphy. Grunberg spotted many errors, made many valuable suggestions and is an eagle eyed proofreader. Murphy suggested major revisions in the ordering of my chapters.

I thank Dr. Isaac To and Dr. Nicholas R. Todd for pointing out errors.

I also thank the ever cooperative Alan Bromborsky for extending \mathcal{G} Algebra to make it more useful to the readers of this book.

Thanks again to Professor Kate Martinson for help with the cover design.

In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius.

- C. F. Gauss

Printings

From time to time I issue new printings of this book, with corrections and minor improvements. The printing version is shown on the title page. They are listed below.

Second Printing. I thank again Gregory Grunberg for many suggestions and expert proofreading. Christoph Bader and Dr. Gavin Polhemous pointed out shortcomings in the notation of Section 5.5. And I thank Dr. Manuel Reenders, a recent arrival, for many suggestions and corrections, especially with regard to the exercises and problems.

Third Printing. I thank a new eagle eyed reader, Nicholas H. Okamoto, for sending me errata.

Fifth Printing. I thank the very careful new reader Professor Mark R. Treuden for helpful comments and corrections.

August 2019 Printing. A new Section 5.4, *Exact Fields*, translates differential forms language to geometric calculus language: closed fields, exact fields potentials, etc.. It was gathered and improved from existing sections.

May 2020 Printing. The idea of a tangent map has been moved to a more appropriate place. There are a few new exercises/problems. All errors known to me have been corrected. All were minor.

October 2020 Printing. I've added material on the Helmholtz decomposition. The last part of Section 10.4 has been moved to Section 10.5. The section also contains some recently published results about antiderivatives. The new Section 11.4 introduces the differential geometry of manifolds of arbitrary dimension. There are several other small improvements.

January 2021 Printing. There are minor improvements.

June 2021 Printing. Eq. (9.3) is new. It provides a better understanding of the definitions of the surface and flux integrals. The last two sections of Chapter 10 have been rearranged. A geometric calculus version of the Helmholtz decomposition has been added to Section 10.5 to go with the vector calculus version in Section 5.4. There is a new Section 11.4, Manifolds in \mathbb{R}^n . There are many minor improvements and corrections.

January 2022 Printing. I have tried to make the text clearer in a number of places. Section 10.5, Analytic Functions, has been rearranged yet again. All errors/typos known to me have been corrected.

The text is improved in several places. The definition of limit have been given a new pictorial form, enabling better understanding of this concept. All errors/typos known to me have been corrected.

September 2023 Printing. There is a new short description of the gradient descent algorithm. All errors/typos known to me have been corrected.

To the Student

Appendix A is a review of some items from *Linear and Geometric Algebra* (LAGA) used in this book. A quick read through it might be helpful before starting this book.

I repeat here my advice from *Linear and Geometric Algebra*.

Research clearly shows that *actively* engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

- Don't just read the text, *study* the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you will understand.
- Instructors in your previous mathematics courses have probably urged you to try to *understand*, rather than simply memorize. That advice is especially appropriate for this text.
- Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand while you read.
- Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.
- Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.
- Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Some are trivial, there just to make sure that you are paying attention. But do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!

The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that “mathematical frame of mind”.

Write your solutions neatly in clear correct English.

- Proofs are important, but perhaps less so than the above. On a first reading, don't get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.
- Important: take the above points seriously!

The World Wide Web makes it possible for me to leave out material that I would otherwise have to include. For example, the book refers to the *Coulomb force* without defining it. Perhaps you already know what it is. If not, and you want to know, actively engage the course material: Google it.

Index

- $B - A$, 14
 $D_{\mathbf{h}}\mathbf{f}$, 33
 $Hf(\mathbf{x})$, 95
 K , 173
 S , 131
 \mathbf{b} , 163
 $\bar{\mathbf{n}}$, 143
 ∇ , 60
 ∇^2 , 62
 $\nabla_{\mathbf{h}}\mathbf{f}$, 33
 ∂ , 87
 $\dot{\nabla}$, 61
 \mathbf{e}_r , 82
 $\mathbf{f}'_{\mathbf{p}}$, 89
 $\mathbf{f}'_{\mathbf{x}}$, 29
 $\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$, 127
 $\hat{\mathbf{w}}_k$, 82
 $\mathbf{i}, \mathbf{j}, \mathbf{k}$, 4
 $\iiint_V F dV$, 125
 $\iint_A F dA$, 121
 $\iint_S dS F$, 132
 $\iint_S d\mathbf{S} F$, 134
 $\iint_S d\sigma F$, 135
 $\int_C F ds$, 110
 $\int_a^b f dx$, 106
 κ , 163
 κ_g , 177
 κ_n , 177
 $\mathcal{L}(\mathbf{U}, \mathbf{V})$, 189
 \mathbf{n} , 163
 ϕ , 115
 ∂M , 141
 $\partial_{\mathbf{h}}\mathbf{f}(\mathbf{p})$, 169
 $\partial_{\mathbf{h}}$, directional derivative
 \mathbb{R}^n , 33
 ∂_i , 23
 ∂_u , 87
 $\partial_{\mathbf{uv}}$, 97
 ∂_{ij} , 40
 $\frac{\partial}{\partial x_i}$, 23
 τ , 164
 $\mathbf{I} \quad \mathbf{I}$, 171
 $\mathbf{T}_{\mathbf{p}}$, 54
 $\hat{\mathbf{B}}$, 131
 $\hat{\mathbf{n}}$, 4, 170
 $\hat{\mathbf{t}}$, 131
 \mathbf{x}_u , 54
 $\{\mathbf{w}^j\}$, 80
 $\{\mathbf{w}_k\}$, 80
 ds , 110
 f_x , 24
 f_{xy} , 26
 \mathcal{G} Algebra, viii
- adjoint, 71, 188
analytic function, 157
angular momentum, 77
antiderivative, 107, 159
arclength, 111
axial vector, 53
- basis
 reciprocal, 85
binormal vector, 163, 164
boundary, 142
boundary values, 158
bounded function, 106
bounded set, 94
- Cauchy's integral formula, 158
Cauchy's integral theorem, 157
Cauchy-Riemann equations, 157
Cauchy-Pompeiu formula, 158
central field, 77-79
chain rule, 35
change of variables, 127
circulation, 115, 117, 153

- closed
 - curve, 118
 - interval, 14
 - set, 14
- commutator, 52, 184
- compact set, 94
- complement, 14
- congruent, 165
- connected set, 20
- conservation
 - angular momentum, 77
 - energy, 77
- conservation law, 76, 78
- conservative, 72
- conservative field, 73, 77
- conserved quantity, 76
- continuity equation, 150
- continuous function, 18
- continuously differentiable, 30
 - well-defined, 92
- contractible, 73, 75
 - closed curve, 73
- coordinate independent
 - ∇ , 60
 - curl, 62
 - curvilinear, 81
 - divergence, 62
- coordinates
 - orthogonal, 82
- Coulomb force, 75
- covariant derivative, 92, 184
- cross product, 55, 188
- curl, 62, 69, 70, 117
- curl theorem, 151
- curvature, 163, 184
- curve
 - parameterization, 5
- curvilinear coordinates, 11, 80
- cylindrical coordinates, 11, 82

- Darboux basis, 177
- Darboux bivector, 165
- De Morgan's laws, 14
- derivative
 - covariant, 92, 184
 - differential, 28
 - directional, 33, 67, 169, 184
 - gradient, 60
 - partial, 23
 - vector, 141
- derivative test
 - first, 93, 94
 - second, 93, 95
- determinant, 189
- differentiable, 28, 60
- differential, 28
 - surface, 89
- differential forms, vii, 184, 192
- differential geometry, 163
- directed integral, 135, 143
- divergence, 62, 69, 136, 148
- divergence theorem, 147
- divergence-free, 150
- dot notation, 61
- double integral, 121
- dual, 188
- duality, 188

- Einstein tensor, 184
- elasticity, 27
- electromagnetic field, 66
- electromagnetism, 65
- embedded manifold, 181
- entropy, 101
- equation of continuity, 150
- Euler characteristic, 182
- exact
 - bivector field, 76
 - differential equation, 79
 - field, 72
 - vector field, 73
- extend
 - field on manifold, 49, 194
 - parameter function, 49, 194
- extend F , 49
- exterior derivative, 192
- extrema, 93
- extrinsic, 175
- extrinsic curvature, 184

- field, 59
 - central, 77, 79
 - closed, 72
 - inverse square, 75
- field equation, 184
- first derivative test, 93, 94
- first fundamental form, 181
- fluid, 136
- fluids, 115, 150
- flux, 136, 148

- flux integral, 134
- formulas, 190
- Frenet basis, 164
- Frenet-Serret equations, 164
- fundamental identities, 187
- fundamental theorem
 - scalar calculus, 107
 - geometric calculus, 141
- fundamental theorem of vector calculus, 79

- Gauss map, 173
- Gauss' theorem, 147
- Gauss-Bonnet theorem, 182
- Gaussian curvature, 173
- general relativity, 184
- geodesic, 180
- geodesic curvature, 177
- geodesic normal vector, 177
- geometric calculus, 3
- geometric product, 187
- GPS, 46
- gradient, 59, 60, 67, 68
 - and linear transformations, 71
 - curvilinear coordinates, 81
 - linear transformations, 71
- gradient descent, 94
- gradient theorem, 117, 156
- Green's functions, 158
- Green's identities, 149
- Green's theorem, 154

- harmonic function, 79, 157
- Helmholtz decomposition, 79, 159
- Hessian matrix, 95
- homogeneous function, 38

- ideal gas law, 39
- implicit differentiation, 44
- incompressible, 70
- indefinite integral, 107
- index lowering, 171
- index raising, 171
- infinitesimal, 108
- inflection point, 93
- inner product, 187
- integrable, 106
- integral
 - definite, 106
 - directed, 135, 141
 - double, 121
 - flux, 134
 - how to think about, 108
 - iterated, 123
 - line, 114
 - path, 110
 - scalar, 105, 106
 - surface, 132
 - triple, 125
- integrand, 108
- integration by parts, 148, 153
- intermediate value theorem
 - $f(\mathbf{x})$, 20
 - double integral, 122
- intrinsic, 175
- intrinsic curvature, 184
- inverse function theorem, 42
- irrotational, 70
- iterated integral, 123

- Jacobian, 28
- Jacobian determinant, 28

- Kepler's laws, 78
- kinetic energy, 77

- Lagrange form, 40
- Lagrange multiplier, 98
- Laplacian, 62
- least squares, 97
- level
 - set, 68
- level curve, 7
- level surface, 69
- limit, 15
- line integral, 114
- linear transformation, 188
- Liouville's theorem, 79
- local inverse, 43
- local minimum, 93, 94
 - strict, 93, 94
- longitudinal, 70

- Möbius strip, 88
- manifold, 47, 141
 - with boundary, 47, 141
- Maxwell relation, 26
- Maxwell's equation, 66
- Maxwell's equations, 65, 76
 - integral form, 149

mean value theorem
 scalar, 27
 vector, 39
 measure zero, 106
 meridians, 182
 metric, 113, 168
 mixed partial derivative, 25
 monogenic, 157
 multiple integral, 126
 multivariable calculus, 3

 neighborhood, 13
 Newton's law of gravitation, 75, 78
 Newton's second law, 77, 78
 Noether's theorem, 78
 norm, 188
 normal curvature, 177
 normal plane, 177, 178
 normal section, 177, 178
 normal vector
 $f(\mathbf{x}) = k$ representation, 68
 $z = f(x, y)$ representation, 55
 outward boundary, 143
 principal, 164
 to curve, 163
 to surface, 143
 notation, 4

 open
 interval, 14
 set, 13
 operator
 gradient, 60
 vector derivative, 87
 operator norm, 189
 orientable, 88, 143
 orientation
 manifold, 143
 orthogonal complement
 \mathbb{G}^n , 188
 orthogonal coordinates, 82
 osculating plane, 165
 outer product, 187
 outermorphism, 188
 outward normal, 143
 outward normal vector, 143

 parallels, 182
 parameter, 5
 parameter independent
 T_p , 56
 ds^2 , 169
 continuously differentiable, 87
 parameterize
 arclength, 112
 curve, 5
 surface, 8
 partial derivative, 23
 geometric interpretation, 24
 partial differential equation, 38
 partition, 105
 interval, 105
 path independent, 117, 118
 path integral, 110
 pitch, 112
 planimeter, 154
 Poincaré Lemma, 75
 polar coordinates, 11
 positive definite matrix, 95
 potential, 72
 scalar, 73
 vector, 76
 potential energy, 77
 principal curvatures, 178
 principal vectors, 178
 principle of maximum entropy, 101
 pseudo-Riemannian manifold, 184
 pseudoscalar, 188
 pseudosphere, 173
 pseudovector, 53, 65
 pullback, 91
 pushforward, 91

 reciprocal basis, 81, 85, 85
 relativity
 general, 169, 184
 special, 55, 66
 reverse, 188
 Ricci curvature, 184
 Riemann curvature, 184
 Riemann sum, 105
 Riemannian manifold, 183, 184
 rotational velocity, 52, 53, 165
 rotational velocity bivector, 52
 rotational velocity vector, 53

 saddle point, 95
 scalar calculus, 3
 scalar curvature, 184
 scalar functions, 3

second derivative test, 93, 95
 second fundamental form, 181
 Seifert surface, 152
 shape, 184
 shape operator, 170, 183
 simply connected, 73
 Simpson's rule, 108
 solenoidal, 72
 spacetime, 66, 184
 spherical coordinates, 12, 83
 right handed, 82
 Stokes's theorem, 151
 strict local minimum, 93, 94
 summation convention, 4
 surface
 orientable, 88
 parameterization, 8
 surface integral, 132
 surface of revolution, 133, 173, 176, 182
 symmetry, 78
 SymPy, viii

tangent algebra, 54
 tangent field, 91, 170
 tangent map, 90
 tangent space, 49
 curve, 50
 manifold, 56
 surface, 54
 tangent vector, 56
 curve, 50
 surface, 54
 unit, 112

Tau manifesto, 158
 Taylor series, 41
 Taylor's formula, 40
 telescoping sum, 30
 tensors, vii
 topological invariant, 182
 Torricelli's trumpet, 133
 torsion, 164
 total curvature, 184
 total derivative, 37
 trace, 71
 transverse, 70
 trapezoidal rule, 108
 triple integral, 125
 twisted cubic, 167

uncertainty, 101

vector calculus, 3
 vector derivative, 87
 vector potential, 76

wave equation, 38, 66
 Weingarten equation, 171
 well-defined
 continuously differentiable, 92
 well-defined
 ∇ , 60
 δ , 87
 Wolfram Integrator, 108
 work, 115