Vector and Geometric Calculus

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Geometry without algebra is dumb! - Algebra without geometry is blind!

- David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.

- Allan McRobie and Joan Lasenby

To Ellen

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Preface

This text, *Vector and Geometric Calculus*, is intended for the second year vector calculus course. It is a sequel to my text *Linear and Geometric Algebra.* That text is a prerequisite for this one.

Linear algebra and vector calculus have provided the basic vocabulary of mathematics in dimensions greater than one for the past one hundred years. Geometric algebra generalizes linear algebra in powerful ways. Similarly, geometric calculus generalizes vector calculus in powerful ways.

Traditional vector calculus topics are covered here, as they must be, since readers will encounter them in other texts and out in the world.

The final chapter is a brief introduction to (mostly 3D) differential geometry, used today in many disciplines, including architecture, computer graphics, computer vision, econometrics, engineering, geology, image processing, and physics.

Vector algebra represents a plane in $\mathbb{R}^3$ with a vector orthogonal to the plane (a trick from the point of view of geometric algebra). But it cannot represent planes in higher dimensions. Nor can it represent higher dimensional analogs. This is a serious limitation of vector algebra. Geometric algebra represents the objects with multivectors.

Tensors and differential forms are two formalisms used to extend vector calculus to higher dimensions. Geometric calculus provides an at once simpler, more general, more powerful, and easier to grasp way to break loose from $\mathbb{R}^3$.[1] Section 5.4, *Exact Fields*, translates elementary differential forms definitions, theorems, and examples to geometric calculus.

Linear algebra is the natural mathematical background for vector calculus. Yet even today it is unusual for a vector calculus text to have a linear algebra prerequisite. This has to do, I suppose, with publishers insisting that authors write to the largest possible audience. I cite my text *Linear and Geometric Algebra* freely and pervasively to advantage.

[1] D. Hestenes and G. Sobczyk have argued in detail the superiority of geometric calculus over differential forms (*Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht Holland 1984, Section 6.4, especially at the end.

Vector and geometric algebra and also differential vector and geometric calculus (Part II of this book) are excellent places to help students better understand and appreciate rigor. But for integral calculus (Part III) rigorous proofs at the level of this book are mostly impossible. So I do not try.

Instead, I use the language of infinitesimals, while making it clear that they do not exist within the real number system. I believe that the first and most important way to understand integrals is intuitively: they "add infinitely many infinitesimal parts to give a whole".

Others endorse this approach: "An approach based on [infinitesimals] closely reflects the way most scientists and engineers successfully use calculus."[2] From Lagrange: "When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results, ... we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.[3] And even Cauchy: "My main aim has been to *reconcile* the rigor which I have made a law in my Cours d'Analyse, with the simplicity that comes from the direct consideration of infinitely small quantities"[4] (Emphasis added.)

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that students attempt them while reading the text. That way they immediately confront the concept and get feedback on their understanding. There are also more challenging problems at the end of most sections – almost 200 in all.

The exercises replace the "worked examples" common in most mathematical texts, which serve as "templates" for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Some exercises and problems require the use of the free multiplatform Python module $\mathcal{GA}$lgebra. It is based on the Python symbolic computer algebra library SymPy (Symbolic Python). The file GAlgebraPrimer.pdf describes the installation and use of the module. It is available at the book's web site.

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file "VAGC Instructor.pdf" at the book's web site. Take it for what it is worth.

The first part of the index is a symbol index.

Some material which is difficult or less important is printed in this smaller font.

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[2]Tevian Dray and Corinne Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, The College Mathematics Journal **34**, 283-290 (2003).

[3]Méchanique Analytique, Preface; Ouveres, t. 2 (Paris, 1988), p. 14.

[4]Quoted in *Cauchy's Continuum*, Karin Katz and Mikhail Katz, Perspectives on Science **19**, 426-452. Also at arXiv:1108.4201v2.

There are several appendices. Appendix A reviews some parts of *Linear and Geometric Algebra* used in this book. Appendix B provides a list of some geometric calculus formulas from this book. Appendix C provides a short comparison of differential forms and geometric calculus. Appendix D provides some technical results.

Numbered references to theorems, figures, etc. preceded by "LAGA" are to *Linear and Geometric Algebra*.

There are several URL's in the text. To save you typing, I have put them in a file "URLs.txt" at the book's web site.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book's web site as appropriate.

Acknowledgements. I thank Dr. Eric Chisolm, Greg Grunberg, Professor Philip Kuntz, James Murphy, and Professor John Synowiec for reading all/most of the text and providing and helpful comments and advice. Professor Mike Taylor answered several questions. I give special thanks to Greg Grunberg and James Murphy. Grunberg spotted many errors, made many valuable suggestions and is an eagle eyed proofreader. Murphy suggested major revisions in the ordering of my chapters.

I also thank the ever cooperative Alan Bromborsky for extending $\mathcal{GA}$lgebra to make it more useful to the readers of this book.

Thanks again to Professor Kate Martinson for help with the cover design.

> In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, ... Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius.
>
> - C. F. Gauss

Printings

From time to time I issue new printings of this book, with corrections and minor improvements. The printing version is shown on the title page. I list below those with significant changes or with new helpful readers.

Second Printing. I thank again Gregory Grunberg for many suggestions and expert proofreading. Christoph Bader and Dr. Gavin Polhemous pointed out shortcomings in the notation of Section 5.5. And I thank Dr. Manuel Reenders, a recent arrival, for many suggestions and corrections, especially with regard to the exercises and problems.

Third Printing. I thank a new eagle eyed reader, Nicholas H. Okamoto, for sending me errata.

Fifth Printing. I thank the very careful new reader Professor Mark R. Treuden for helpful comments and corrections.

August 2019 Printing. A new Section 5.4, *Exact Fields*, translates differential forms language to geometric calculus language: closed fields, exact fields potentials, etc.. It was gathered and improved from existing sections.

May 2020 Printing. The notion of a tangent map has been moved to a more appropriate place. There are a few new exercises/problems. All errors known to me have been corrected. All were minor.

October 2020 Printing. I've added material on the Helmholtz decomposition. The last part of Section 10.4 has been split off into a new Section 10.5. The new section also contains some recently published results about antiderivatives. The new Section 11.4 introduces the differential geometry of manifolds of arbitrary dimension. There are several other small improvements.

January 2021 Printing. There are minor improvements.

To the Student

Appendix A is a review of some items from *Linear and Geometric Algebra* (LAGA) used in this book. A quick read through it might be helpful before starting this book.

I repeat here my advice from *Linear and Geometric Algebra*.

Research clearly shows that *actively* engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

- ~~Read~~ Study the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you will understand.

- Instructors in your previous mathematics courses have probably urged you to try to *understand*, rather than simply memorize. That advice is especially appropriate for this text.

- Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand while you read.

- Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.

- Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.

- Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Some are trivial, there just to make sure that you are paying attention. But do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!

The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that "mathematical frame of mind".

Write your solutions neatly in clear correct English.

- Proofs are important, but perhaps less so than the above. On a first reading, don't get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.

- Take the above points seriously!

The World Wide Web makes it possible for me to leave out material that I would otherwise have to include. For example, the book refers to the *Coulomb force* without defining it. Perhaps you already know what it is. If not, and you want to know, actively engage the course material: Google it.

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