LUTHER COLLEGE

Howard Hughes Medical Institute
2016 Math Inquiry Workshop

Euler's Method

It is common in many projects to develop a model for the dependent variable using principles and assumptions about the *rate* at which the dependent variable changes with respect to one or more of the independent variables. For example, it is often assumed that the rate of change of the velocity of a free-falling object is constant. This assumption translates to the equation

$$\frac{dv}{dt} = g$$

where velocity v is the **dependent variable**, time t is the **independent variable**, and the constant g represents the acceleration due to gravity. Equations of this type that include the derivative of the dependent variable are called (ordinary) **differential equations** (ODEs).

Typically, the next step in the process is to use this rate of change relationship to determine a formula $v(t)$ for the dependent variable. In many cases, certainly not all, there are applicable "analytic" techniques to actually find such a formula providing the ability to determine the dependent variable for all values of the independent variable. For more complicated ODEs, or for expediency sake, "numerical" methods are used to accurately *approximate* the dependent variable at a finite number of discrete values of the independent variable.

Euler's method is a rather simple way, and often accurate enough, for numerically integrating, or "solving", an ordinary differential equation of the form

$$\frac{dx}{dt} = f(t)$$

when we are given an **initial condition** (starting value) of x, $x(t_0) = x_0$. The objective is to "construct" the graph of the unknown function $x(t)$ from the known information that includes the graph's initial point (t_0, x_0), the formula $f(t)$ that gives the slope of the graph of $x(t)$ at any value t, and prescribed step-size Δt along the t axis.

Figure 1 shows the actual graph of the function $x(t)$ even though in a true application of Euler's method this graph is not known. It is shown here for the sake of illustration only. The initial point (t_0, x_0) is shown on Figure 1 in black. The slope of the graph of $x(t)$ for $t = t_0$ is shown as a red line. This represents a segment of the line tangent to the graph of $x(t)$ at the point (t_0, x_0). The slope of this line is known by calculating it from the formula, or function, $f(t)$.

Starting with the initial point (t_0, x_0) on the graph, the next step is to determine the coordinates of the point (t_1, x_1), the second black dot on the graph of $x(t)$ shown in Figure 1. The "step" in the t direction to t_1 is the prescribed length of Δt. That is, $t_1 = t_0 + \Delta t$, so the value t_1 is easily determined. The slope of the tangent line will be used to determine (or estimate) x_1, Because the tangent is a line, its slope may also be represented by the ratio $\Delta x/\Delta t$, the "rise over the run." Therefore,

$$\frac{\Delta x}{\Delta t} = f(t_0)$$

This equation is solved for Δx, that is $\Delta x = f(t_0)\Delta t$. So, $x_1 \approx x_0 + \Delta x = x_0 + f(t_0)\Delta t = \tilde{x}_1$. As shown in Figure 1, the value for x_1 found in this way will be an approximation, and a different symbol such as $\tilde{x}_1$ will be used to represent this Euler approximation. In summary, the first "Euler" point $(t_1, \tilde{x}_1)$ is found from the initial point (t_0, x_0) using

$$t_1 = t_0 + \Delta t \tag{1}$$

$$\tilde{x}_1 = x_0 + \Delta x = x_0 + f(t_0)\Delta t \tag{2}$$

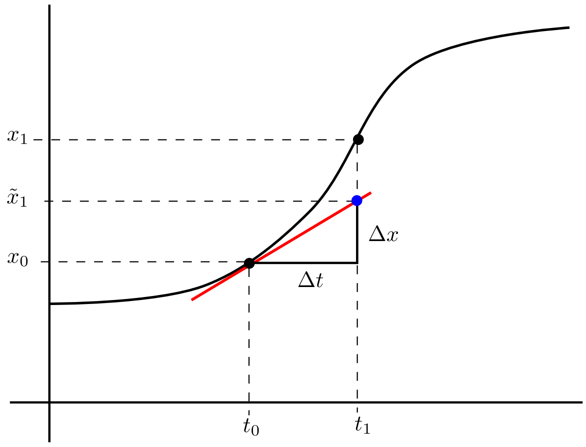

Figure 1: Euler's method

Now, the first Euler point $(t_1, \tilde{x}_1)$ becomes the "new" initial point, and the Euler process is used to calculate the second Euler point $(t_2, \tilde{x}_2)$. That is,

$$t_2 = t_1 + \Delta t \tag{3}$$

$$\tilde{x}_2 = \tilde{x}_1 + \Delta x = \tilde{x}_1 + f(t_1)\Delta t \tag{4}$$

This formulation is carried out a fixed number of times to create a list of Euler points that is used to approximate the unknown curve. The general recursion formula is

$$t_{i+1} = t_i + \Delta t \tag{5}$$

$$\tilde{x}_{i+1} = \tilde{x}_i + f(t_i)\Delta t \tag{6}$$

Figure 2 shows the final result for the Euler method. The unknown curve is shown in the light dotted line. The black points on the dotted curve represent the points the Euler method attempted to locate. The blue dots show the Euler approximation to the actual points. The distance between the two corresponding points represents the error in the Euler approximation. The error is shown for the fourth Euler point in the figure. Note that this difference is entirely along the x axis because the t locations are prescribed exactly in the process.

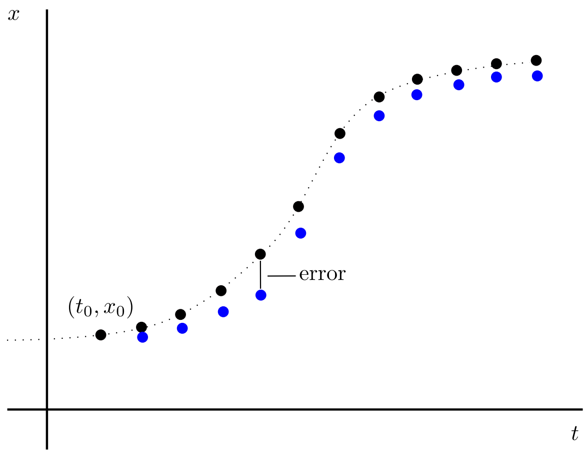

Figure 2: Euler's method

Questions:

1. As depicted in Figure 1, the Euler approximation $\tilde{x}_1$ is an underestimation for the actual value x_1. Can you explain why this is the case?

2. Figure 2 indicates that the error initially grows larger with each new step, but then begins to decrease over a few of the "middle" steps. Why is this true?

3. What characteristic of the graph of the function $x(t)$ determines the accuracy of Euler's method? That is, the accuracy of Euler's method decreases as the ______________ of the graph of x ______________.

Exercises

Make a copy of the spreadsheet **Euler's Method** to complete the following exercises.

1. Solve the following initial value problem using your copy of the Euler's spreadsheet.

$$\begin{cases} \dfrac{dx}{dt} & = & 2t \\ x(0) & = & 1.0 \end{cases}$$

on the interval $0 \leq t \leq 2$.

   (a) Use a step size of $\Delta t = 0.1$.
   (b) Use a step size of $\Delta t = 0.05$.
   (c) The true value graph in this case is given by $x(t) = t^2 + 1$. Find the error in $\tilde{x}(2)$ in both cases above. Are they under- or over-estimations of the true value? Next, compare their values, especially the relative size, or ratio, of the $\Delta t = 0.05$ error to the $\Delta t = 0.1$ error. What do you expect the error to be if the step size is $\Delta t = 0.025$?

2. Use a spreadsheet and Euler's method to solve the following initial value problem.

$$\begin{cases} \frac{dx}{dt} & = & t^2 + 2t \\ x(0) & = & 1.0 \end{cases}$$

on the interval $0 \leq t \leq 2$ and an initial step size of $\Delta t = 0.1$. Determine the value of $x(2)$ accurate to 2 decimal places.

3. Use a spreadsheet and Euler's method to solve the following initial value problem.

$$\begin{cases} \frac{dx}{dt} & = & \cos t \\ x(0) & = & 1.0 \end{cases}$$

on the interval $0 \leq t \leq 2\pi$ and a step size of $\Delta t = \pi/10$.

4. In more general initial value problems, the slope of the curve that is being constructed depends on both the t values <u>and</u> the x value. The initial value problem in this case is represented by

$$\begin{cases} \frac{dx}{dt} & = & f(t, x) \\ x(t_0) & = & x_0 \end{cases}$$

Euler's method can be applied to solves such a case. Modify your Euler spreadsheet to solve the following problem

$$\begin{cases} \frac{dx}{dt} & = & t + x \\ x(0) & = & 1.0 \end{cases}$$

on the interval $0 \leq t \leq 2$ and a step size of $\Delta t = 0.05$.