Math 141A   Review Exercises Solutions   Spring 2014

1. Given $f(x) = xg(x^2)$, $f'(x) = g(x^2) + xg'(x^2)2x = g(x^2) + 2x^2g'(x^2)$

$f''(x) = g'(x^2)\cdot 2x + 4xg'(x^2) + 2x^2g''(x^2)2x$

$= \boxed{6xg'(x^2) + 4x^3g''(x^2)}$

2. $y' = -2\sin 2x$, $y'' = -2^2\cos 2x$, $y^{(3)}(x) = 2^3\sin 2x$, $y^{(4)}(x) = 2^4\cos 2x$

so, $y^{(48)}(x) = 2^{48}\cos 2x$ $\boxed{y^{(50)}(x) = -2^{50}\cos 2x}$

3. $y - y_0 = m(x - x_0)$. Here $(x_0, y_0) = (1,1)$, and $m = \left.\dfrac{dy}{dx}\right|_{x=1, y=1}$

$2x + y + xy' + 2y' = 0 \Rightarrow 2 + 1 + y' + 2y' = 0$

$\Rightarrow y' = -3/3 = -1$

$\boxed{(y-1) = -1(x-1)}$

4. $f(x) = 2x^3 + 3x^2 - 36x$, $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6)$

$= 6(x+3)(x-2)$

y'': $-$ 0 $+$

y': $+$ 0 $-$ 0 $+$

-3 $-\frac{1}{2}$ 0 2

(a) inc: $(-\infty, -3)$ $(2, \infty)$

dec $(-3, 2)$

(b) loc max $(-3, 81)$, local min $(2, -44)$

(c) $f''(x) = 12x + 6$ $f''(x) = 0 \Rightarrow x = -\frac{1}{2}$

cc↓ $(-\infty, -\frac{1}{2})$ cc↑ $(\frac{1}{2}, \infty)$

(d) IP: $(-\frac{1}{2}, f(-\frac{1}{2})) = (-\frac{1}{2}, -\frac{1}{4} + \frac{3}{4} + 18) = (-\frac{1}{2}, \frac{37}{2})$

(e) no absolute extremes

4. (ii) $f(x) = \sin x + \cos x$, note that f is 2π-periodic so need only consider the interval $[0, 2\pi]$

$f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$

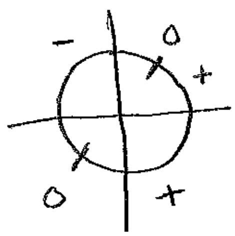
$f'(x) = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

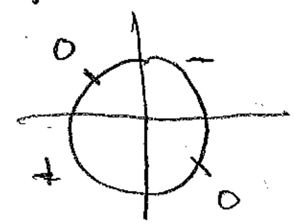
$f''(x) = 0 \Rightarrow \sin x = -\cos x$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

y'



y''



(a) inc $\left(-\frac{3\pi}{4} + 2n\pi, \frac{\pi}{4} + 2n\pi\right)$

dec $\left(\frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi\right)$

(b) loc max $\left(\frac{\pi}{4} + 2n\pi, \sqrt{2}\right)$

loc min $\left(\frac{5\pi}{4} + 2n\pi, -\sqrt{2}\right)$

(c) CC↑ $\left(\frac{3\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi\right)$

CC↓ $\left(-\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi\right)$

(d) IPs $\left(\frac{3\pi}{4} + 2n\pi, 0\right)$

$\left(\frac{7\pi}{4} + 2n\pi, 0\right)$

(e) Abs max $\left(\frac{\pi}{4} + 2n\pi, \sqrt{2}\right)$

Abs min $\left(\frac{5\pi}{4} + 2n\pi, -\sqrt{2}\right)$

5. $y' = 3x^2 + 1$, $y' = 0 \Rightarrow 3x^2 = -1$

so $y' > 0 \; \forall x$

$y'' = 6x$ $\quad y'' = 0 \Rightarrow x = 0$

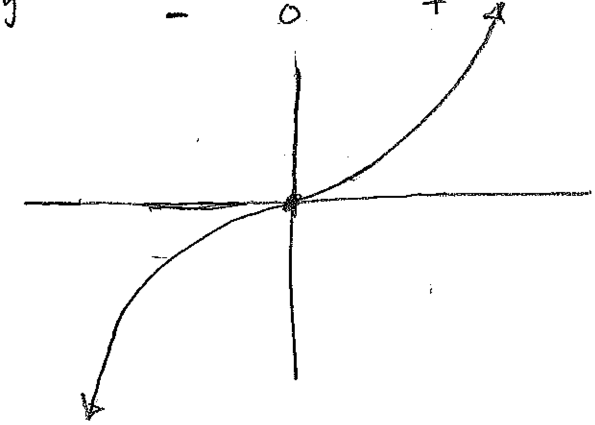
x intercepts: $x = 0$

y intercept: $y = 0$

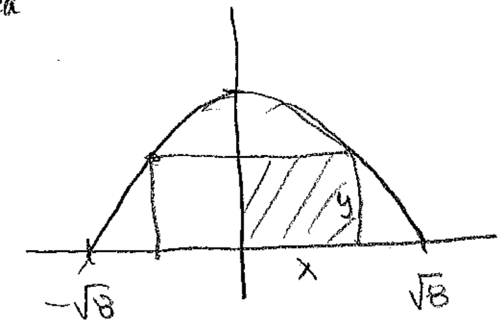
no local extremes / abs extremes

IP $(0,0)$

y'' - 0 +



6. By symmetry, will maximize area given by $A = xy$, which will maximize the area of the whole rectangle.



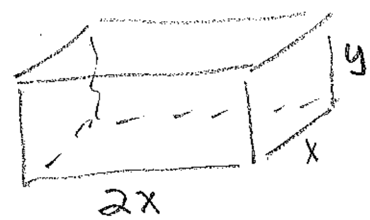
$A = xy = x(8 - x^2) = 8x - x^3$

domain $[0, \sqrt{8}]$

cps: $A'(x) = 8 - 3x^2$ $\quad A' = 0 \Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \sqrt{\frac{8}{3}}$

Because $A(\sqrt{\frac{8}{3}}) > 0$ ($\leftarrow$ endpt values of A) $x = \sqrt{\frac{8}{3}}$ and $y = 8 - \frac{8}{3} = \frac{16}{3}$ will maximize the small rectangle area. Ans: $\boxed{x = 2\sqrt{\frac{8}{3}},\ y = \frac{16}{3}}$

7. Cost = $\underbrace{\$10 \cdot 2x^2}_{\text{base}} + \underbrace{\$6 \cdot 2 \cdot xy}_{\text{sides}} + \underbrace{\$6 \cdot 2 \cdot 2xy}_{\text{sides}}$



$= 20x^2 + 36xy$

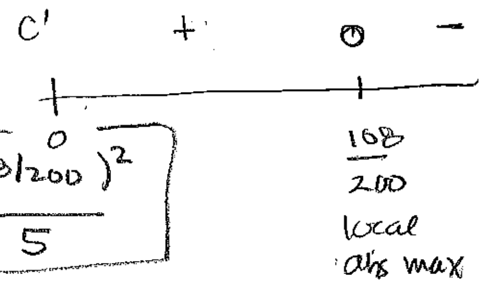
Constraint: $2x^2 y = 10 \Rightarrow y = \frac{x^2}{5}$

$C(x) = 20x^2 + \frac{36x^3}{5}$ $\quad$ Domain $(0, \infty)$

cps: $C'(x) = 40x - \frac{108}{5}x^2 = x\left(40 - \frac{108}{5}x\right)$

$C'(x) = 0 \Rightarrow x = 0$ or $x = \frac{108}{200}$

Based on the sign graph of C' $x = \frac{108}{200}$ gives a local and abs max.



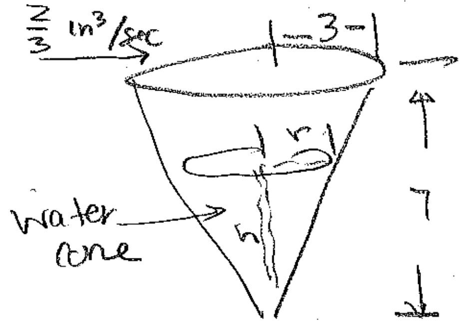
Dimensions: $\boxed{x = \frac{108}{200},\ y = \frac{(108/200)^2}{5}}$

8. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$; $\frac{dr}{dt} = -1 \quad r=2$

$$\Rightarrow \frac{dV}{dt} = 4\pi(2)^2(-1) = \boxed{-16\pi \ \frac{\text{in}^3}{\text{hr}}}$$

9. Variables:

V = volume of water cone
h = height of water cone
r = radius of water cone



Rates

$\frac{dV}{dt} = \frac{2}{3}$ in³/sec

$\frac{dh}{dt}$ = find, $h=4$, $\frac{dr}{dt} = x$ (don't know, don't need)

Relate the variables

Similar Δ's $\quad \frac{r}{h} = \frac{3}{7}$

$r = \frac{3}{7}h$

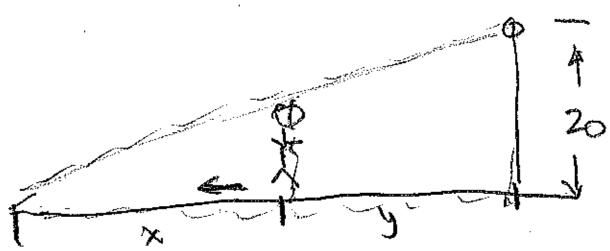
$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{3}{7}h\right)^2 h = \frac{1}{3}\cdot\frac{9}{49}\pi h^3$

$\frac{dV}{dt} = \frac{9}{49}\pi h^2 \frac{dh}{dt} \Rightarrow \frac{9}{49}\pi(4)^2\frac{dh}{dt} = \frac{2}{3} \Rightarrow \frac{dh}{dt} = \frac{\cancel{2}}{3}\cdot\frac{1}{\pi}\,\frac{49}{9\cdot\cancel{16}_{8}}$

$= \boxed{\frac{49}{216\pi} \ \text{in/sec}}$

10.



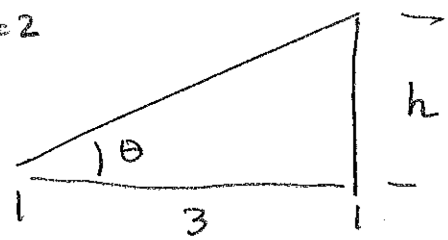
$\frac{dx}{dt} = 2$ ft/sec $\quad \frac{dy}{dt} = ?$ when $y=20$ $\qquad$ Use similar Δ's

$\frac{x+y}{20} = \frac{x}{6} \Rightarrow 6x+6y = 20x \Rightarrow 6y = 14x$

$y = \frac{14}{6}x = \frac{7}{3}x$

$\frac{dy}{dt} = \frac{7}{3}\frac{dx}{dt} = \left(\frac{7}{3}\right)(2) = \boxed{\frac{14}{3} \ \text{ft/sec}}$

11. $\frac{dh}{dt} = 200$ miles/hr , $\frac{d\theta}{dt} = ?$ when $h=2$



$\tan\theta = \frac{h}{3} \Rightarrow \theta = \arctan\frac{h}{3}$

$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{3}\right)^2} \cdot \frac{1}{3}\frac{dh}{dt}$

$= \left(\frac{1}{1+\frac{4}{9}}\right) \cdot \frac{1}{3} \cdot 200 = \frac{\cancel{9}^{3}}{13} \cdot \frac{1}{\cancel{3}} \cdot 200 = \boxed{\frac{600}{13} \ \frac{\text{rad}}{\text{sec}}}$

12. $a(t) = v'(t) = \sin t + \cos t \Rightarrow v(t) = -\cos t + \sin t + C$

$v(0) = 5 \Rightarrow -\cos 0 + \sin 0 + C = 5 \Rightarrow -1 + C = 5 \Rightarrow C = 6.$

$v(t) = s'(t) = -\cos t + \sin t + 6 \Rightarrow s(t) = -\sin t - \cos t + 6t + C$

$s(0) = 0 \Rightarrow -\sin 0 - \cos 0 + 6\cdot 0 + C = 0 \Rightarrow -1 + C = 0$

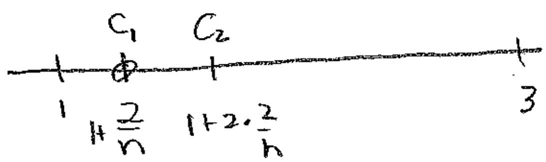
$\Rightarrow C = 1.$ $\boxed{s(t) = -\sin t - \cos t + 6t + 1}$

13. $LS = [7.6 + 6.8 + 6.2 + 5.7 + 5.3]\,2$ = lower estimate

$US = [8.7 + 7.6 + 6.8 + 6.2 + 5.7]\,2$ = upper estimate

14. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + k\frac{2}{n}\right)^2 \frac{2}{n} = \int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = 9 - \frac{1}{3}$

$= \boxed{\frac{26}{3}}$



15. $\text{ave}(f(x)) = \frac{1}{2-0} \int_0^2 [x + \sin(\pi x)]\,dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{1}{\pi}\cos(\pi x)\right]_0^2$

$= \frac{1}{2}\left[2 - \frac{1}{\pi} - \left(0 - \frac{1}{\pi}\right)\right] = \frac{1}{2}[2 + 0] = \boxed{1}$

16. (a) on the graph (Note: $g'(x) = f(x)$, $g'' = f'(x)$ by FTC part I)

(b) increasing where $f>0$

$(0,1)$ $(3,8)$

decreasing where $f<0$

$(1,3)$, $(8,10)$

(c) local max $(1, \tfrac{1}{2})$, $(8, 8\tfrac{1}{2})$

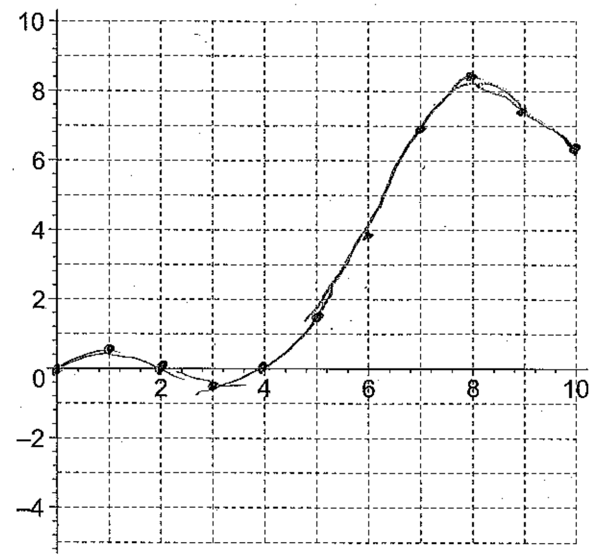
local min $(3, -\tfrac{1}{2})$

global max $(8, 8\tfrac{1}{2})$

global min $(3, -\tfrac{1}{2})$

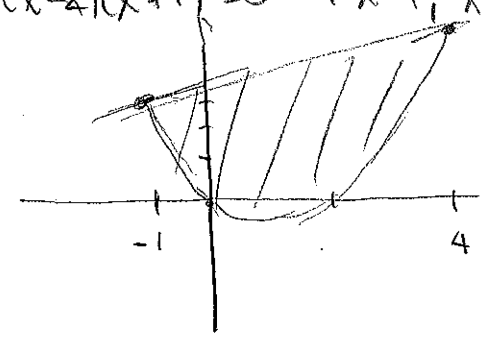
(d) CCU: $(2,6)$, CCD $(0,2)$, $(7,9)$

(e) IPs: $(2,0)$



17. $y=y \Rightarrow x^2-2x = x+4 \Rightarrow x^2-3x-4=0$

$\Rightarrow (x-4)(x+1)=0 \Rightarrow x=4,\ x=-1$



$$A = \int_{-1}^{4} x+4-(x^2-2x)\,dx$$

$$= \int_{-1}^{4} (-x^2+3x+4)\,dx$$

$$= \left[\frac{-x^3}{3} + \frac{3}{2}x^2 + 4x\right]_{-1}^{4}$$

$$= \frac{-64}{3} + \frac{48}{2} + 16 - \left(-\frac{1}{3} + \frac{3}{2} - 4\right)$$

18. $x^2-4x+3=0 \Rightarrow (x-3)(x-1)=0$

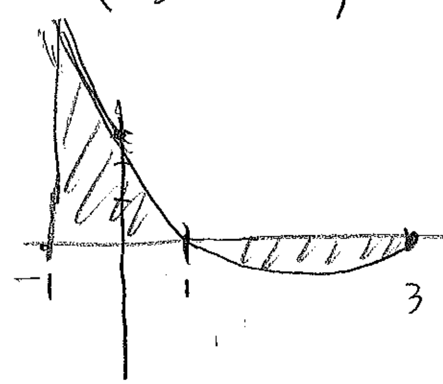
$\Rightarrow x=3$ or $x=1$

$$\int_{-1}^{3} |x^2-4x+3|\,dx = \int_{-1}^{1} (x^2-4x+3)\,dx - \int_{1}^{3} (x^2-4x+3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{-1}^{1} + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{3}^{1}$$

$$= 2\left(\frac{1}{3} - 2 + 3\right) - \left(-\frac{1}{3} - 2 - 3\right) - \left(\frac{27}{3} - 18 + 9\right)$$

$$= \frac{8}{3} + \frac{16}{3} - \cancel{\frac{27}{3}} + \cancel{\frac{27}{3}} = \boxed{\frac{24}{3}}$$



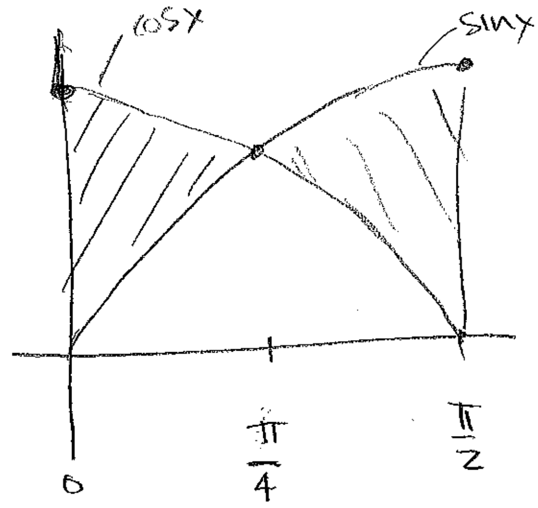
19. $A = \int_0^{\pi/2} |\cos x - \sin x|\, dx$

$= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$

$= 2\int_0^{\pi/4} (\cos x - \sin x)dx$

$= 2\left[+\sin x + \cos x\right]_0^{\pi/4}$

$= 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (\sin 0 + \cos 0)\right] = \boxed{2\sqrt{2} - 2}$

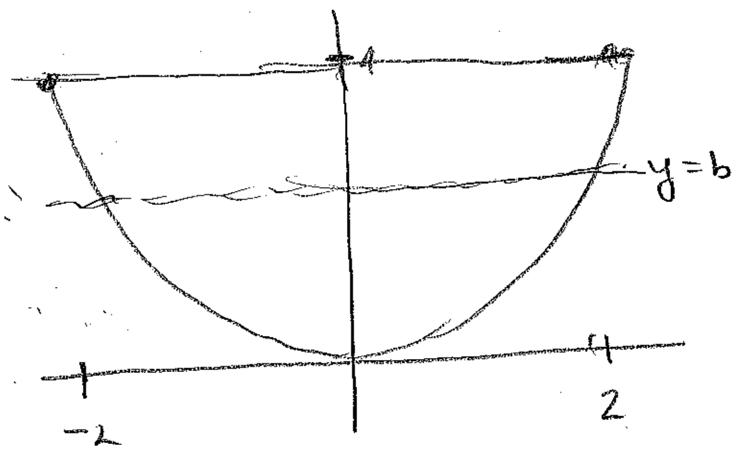


20. A = total area

$= 2\int_0^2 (4 - x^2)dx$

$= 2\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$



Want $2\int_0^2 (b - x^2)dx = \frac{1}{2}\frac{32}{3} \Rightarrow \int_0^2 (b - x^2)dx = \frac{8}{3}$

$\Rightarrow \left[bx - \frac{x^3}{3}\right]_0^2 = \frac{8}{3} \Rightarrow 2b - \frac{8}{3} = \frac{8}{3}$

$\Rightarrow 2b = \frac{16}{3}$

$\Rightarrow b = \boxed{\frac{8}{3}}$